Proof of Goldbach conjecture

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Abstract. This paper is a trial to prove Goldbach conjecture according to the following process.

1. We make the function $l'(n)$ regarding n. When an even number n is divided into 2 odd numbers, $l'(n) \leq l(n)$ holds true.
   $l(n)$: the total number of ways to divide an even number n into 2 prime numbers.
2. We find that $0 < l'(n)$ holds true in $4 * 10^{18} \leq n$.
3. Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$.
4. Goldbach conjecture is true from the above item 1. — 3.

1. Introduction

1.1 When an even number n is divided into 2 odd numbers x and y, we can express the situation as pair (x, y) like the following (1).

$$n = x + y = (x, y) \qquad (n = 6, 8, 10, 12, \cdots\cdots \quad x, y : \text{odd number}) \tag{1}$$

n has $n/2$ pairs like the following (2).

$$(1, n-1), (3, n-3), (5, n-5), \cdots\cdots, (n-5, 5), (n-3, 3), (n-1, 1) \tag{2}$$

We define as follows.

Prime pair : the pair where both x and y in (x, y) are prime numbers
Composite pair : the pair other than the above prime pair
$l(n)$: the total number of the prime pairs which exist in $n/2$ pairs shown by the above (2). (p, q) is regarded as the different pair from (q, p). $(p, q$: prime number$)$

1.2 Goldbach conjecture can be expressed as the following (3).

$$1 \leq l(n) \qquad (n = 6, 8, 10, 12, \cdots\cdots) \tag{3}$$

Goldbach conjecture is confirmed to be true up to $n = 4 * 10^{18}$. So we can try to prove Goldbach conjecture in the following condition.

$$4 * 10^{18} < n \tag{4}$$

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2. The total number of composite pair in $n/2$ pairs

2.1 We can calculate the total number of composite pair where x or y is divisible by p in $n/2$ pairs as follows. (p : prime number)

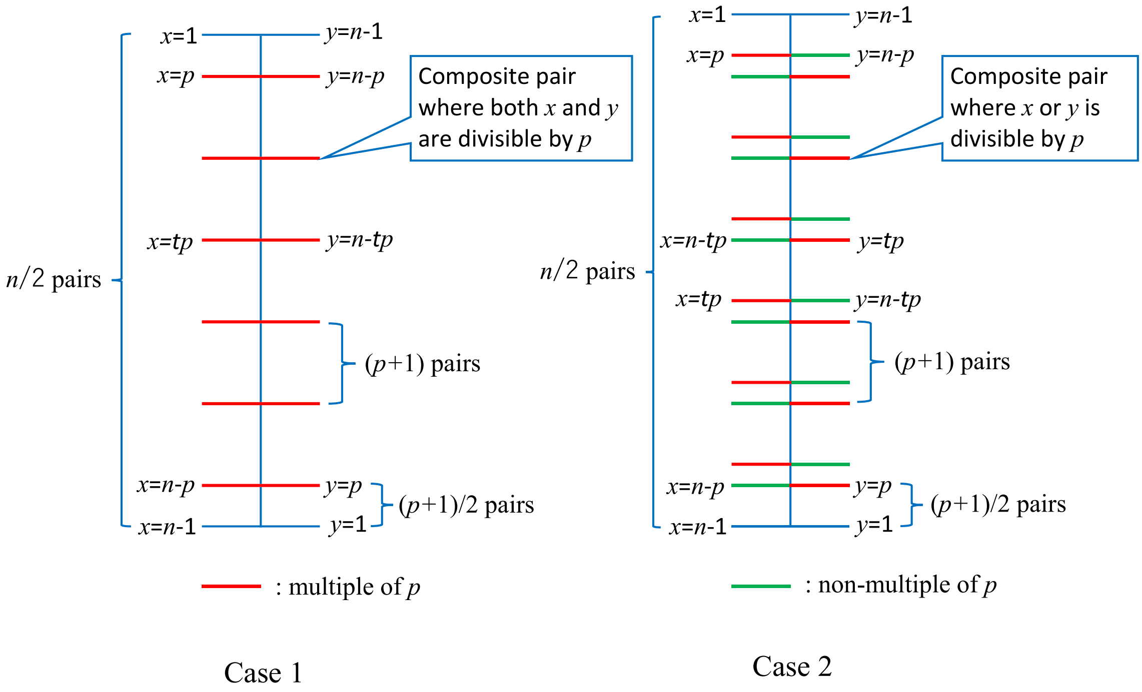


Figure 1 : Case 1 and Case 2

Case 1 : When n is divisible by p, both x and y are divisible by p in the pair of $(x, y) = (tp, n - tp)$. $(t = 1, 3, 5, 7, \cdots\cdots, n/p - 5, n/p - 3, n/p - 1)$ Then $(n/p)/2$ composite pairs where both x and y are divisible by p exist in $n/2$ pairs.

Case 2 : When n is not divisible by p, $(n - tp)$ is not divisible by p in the pair of $(x, y) = (tp, n - tp)$. $(t = 1, 3, 5, 7, \cdots\cdots, 2 * \lfloor (n/2)/p \rfloor - 5, 2 * \lfloor (n/2)/p \rfloor - 3, 2 * \lfloor (n/2)/p \rfloor - 1)$ Then $\lfloor (n/2)/p \rfloor$ composite pairs where x is divisible by p exist in $n/2$ pairs as shown in the following [Memo 1]. If $(n - p)$ is a prime number at $t = 1$, $(p, n - p)$ is a prime pair and $(\lfloor (n/2)/p \rfloor - 1)$ composite pairs exist. Since the case of $(x, y) = (n - tp, tp)$ is also similar, the total number of composite pairs where x or y is divisible by p is $2 * \lfloor (n/2)/p \rfloor$ or $(2 * \lfloor (n/2)/p \rfloor - 2)$. $2 * \lfloor (n/2)/p \rfloor$ and $(2 * \lfloor (n/2)/p \rfloor - 2)$ can be approximated to $\lfloor n/p \rfloor$ because n is a large number as shown in (4).

Memo 1

If we divide $n/2$ odd numbers of $(1,\ 3,\ 5,\ \cdots\cdots, n-1)$ into groups of p, there will always be a multiple of p in the middle of each group as shown in the following (Figure 2). When $0 \leq \mathrm{frac}\{(n/2)/p\} < 0.5$ holds true as shown in Case 3, the total number of multiples of p is $\lfloor (n/2)/p \rfloor$. Frac(x) means $(x - \lfloor x \rfloor)$. When $0.5 \leq \mathrm{frac}\{(n/2)/p\} < 1$ holds true as shown in Case 4, the total number of multiples of p is $\lceil (n/2)/p \rceil$. Then we can say that the total number of multiples of p is $\lfloor (n/2)/p + 0.5 \rfloor$. $\lfloor (n/2)/p + 0.5 \rfloor$ can be approximated to $\lfloor (n/2)/p \rfloor$ because n is a large number as shown in (4).

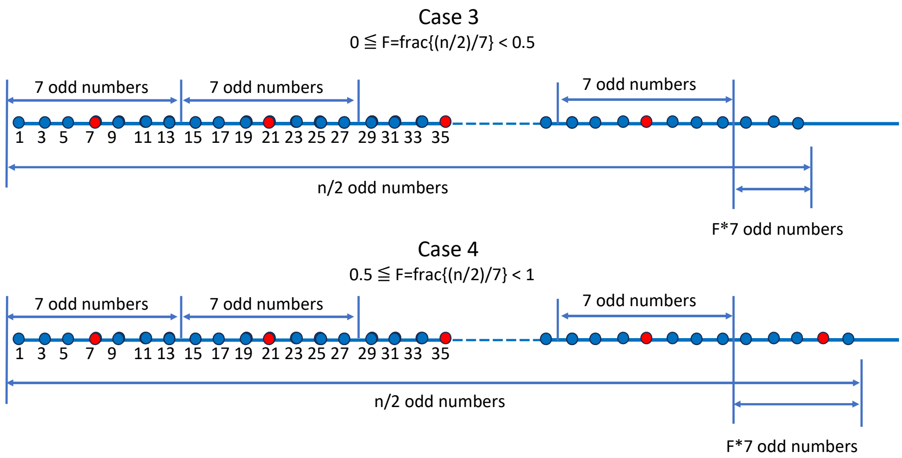


Figure 2 ： Case 3 and Case 4

2.2 We express prime numbers as follows.

$p_1 = 3,\ p_2 = 5,\ p_3 = 7, \cdots\cdots$

$(3,\ 5,\ 7, \cdots\cdots, p_{j-1}, p_j)$ are prime numbers arranged in ascending order. From the above item 2.1 we can calculate $M(n)$: {the total number of composite pair in $n/2$ pairs} as the following (5).

$$M(n) = m(3) + m(5) + m(7) + \cdots + m(p_k) + \cdots + m(p_{j-1}) + m(p_j) + (2,0)_n \quad (k = 1, 2, 3, \cdots\cdots, j-1, j) \tag{5}$$

$m(p_k)$: the total number of pairs where x or y is a composite number and divisible by p_k but not divisible by any prime number of $(3,\ 5,\ 7,\ \cdots\cdots, p_{k-1})$

p_j : the largest prime number that satisfies $p < \sqrt{n}$.

$(2,0)_n$: when $(n-1)$ is a prime number, $(2,0)_n = 2$ holds true because both $(1, n-1)$ and $(n-1, 1)$ are composite pairs. When $(n-1)$ is a composite number, $(2,0)_n = 0$ holds true. If n is large enough, $(2,0)_n$ can be ignored.

We have the following (6) from the above (5).

$$l(n) = n/2 - M(n) \tag{6}$$

2.3 Here we define an imaginary function $M'(n)$ like the following (7).

$$M'(n) = m'(3) + m'(5) + m'(7) + \cdots + m'(p_k) + \cdots + m'(p_{j-1}) + m'(p_j) + (2,0)_n \quad (7)$$

$m'(p_k)$: the total number of pairs where x or y is a composite number and divisible by p_k but not divisible by any prime number of (3, 5, 7, $\cdots\cdots, p_{k-1}$) when we assume that n is not divisible by p_k

We can have the above $M'(n)$ by assuming that n is not divisible by any prime number of (3, 5, 7,$\cdots\cdots, p_{j-1}, p_j$) like $n = 2^b$ or $n = 2*p_c$. $(b = 3, 4, 5, \cdots\cdots \quad c = 1, 2, 3, \cdots\cdots)$ In other words we calculate $m'(p_k)$ by doubling the total number of x which is a composite number and divisible by p_k but not divisible by any prime number of (3, 5, 7,$\cdots\cdots, p_{k-1}$).

We can have the following (8) from item 2.1 and the above difinition of $m(p_k)$ and $m'(p_k)$. We have the following (9) from (8).

$$m(p_k) \leq m'(p_k) \qquad (k = 1, 2, 3, \cdots\cdots, j) \quad (8)$$
$$M(n) \leq M'(n) \quad (9)$$

2.4 We define $l'(n)$ as the following (10) and we have the following (11) from (6), (9) and (10).

$$l'(n) = n/2 - M'(n) \quad (10)$$
$$l'(n) = n/2 - M'(n) \leq n/2 - M(n) = l(n) \quad (11)$$

2.5 We have the following (24) and (25) in [Appendix 1 : Investigation of $l'(n)$].

$$0 < l'(n) \leq l(n) \qquad (4 * 10^{18} \leq n) \quad (24)$$
$$1 \leq l(n) \qquad (4 * 10^{18} \leq n) \quad (25)$$

3. Conclusion

Goldbach conjecture is true from the following item 3.1 and 3.2.

3.1 Goldbach conjecture is true in $4 * 10^{18} \leq n$ from the above (25).

3.2 Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$.

Appendix 1. : Investigation of $l'(n)$

1.1 We can calculate $m'(p_k)$ on the condition that n is not divisible by p_k as follows.

1.1.1 $m'(3)$ can be calculated as the following (12).

$$m'(3) = 2 * \lfloor (n/2)/3 \rfloor - 2 \fallingdotseq 0.33 * n \quad (12)$$

Memo 2

3 is a multiple of 3 but not a composite number. $\lfloor (n/2)/3 \rfloor$ is the total number of multiples of 3 in x or y. Then we must subtract 2 in the above (12). But if n is large enough, (-2) can be ignored. The same applies to the following (13)—(16).

1.1.2 $m'(5)$ can be calculated as the following (13).

$$m'(5) \risingdotseq 2 * \{(n/2) - m'(3)\} * a_2 = 0.068 * n$$
$$a_2 = \frac{(3*5)\{1/5 - 1/(3*5)\}}{(3*5)(1-1/3)} = 1/5 \tag{13}$$

$\{(n/2) - m'(3)\}$ is the total number of pairs where both x and y are not divisible by 3. a_2 is the proportion of multiples of 5 in the total number of odd numbers N which are not divisible by 3 in $1 \leq N \leq (2*3*5-1)$.

1.1.3 $m'(7)$ can be calculated as the following (14).

$$m'(7) \risingdotseq 2 * \{(n/2) - m'(3) - m'(5)\} * a_3 = 0.029 * n$$
$$a_3 = \frac{(3*5*7)\{1/7 - 1/(3*7) - 1/(5*7) + 1/(3*5*7)\}}{(3*5*7)\{1 - 1/3 - 1/5 + 1/(3*5)\}} = 1/7 \tag{14}$$

$\{(n/2) - m'(3) - m'(5)\}$ is the total number of pairs where both x and y are not divisible by either 3 or 5. a_3 is the proportion of multiples of 7 in the total number of odd numbers N which are not divisible by 3 or 5 in $1 \leq N \leq (2*3*5*7-1)$.

1.1.4 Similarly $m'(11)$ can be calculated as the following (15).

$$m'(11) \risingdotseq 2 * \{(n/2) - m'(3) - m'(5) - m'(7)\} * a_4 = 0.013 * n \tag{15}$$

$$a_4 = \frac{(3*5*7*11)\{1/11-1/(3*11)-1/(5*11)-1/(7*11)+1/(3*5*11)+1/(5*7*11)+1/(7*3*11)-2/(3*5*7*11)\}}{(3*5*7*11)\{1-1/3-1/5-1/7+1/(3*5)+1/(5*7)+1/(7*3)-2/(3*5*7)\}} = 1/11$$

1.1.5 $m'(p_k)$ can be calculated as the following (16).

$$m'(p_k) \risingdotseq 2 * \{(n/2) - m'(3) - m'(5) - m'(7) - \cdots\cdots - m'(p_{k-1})\} * a_k$$
$$a_k = a_{kk}/a_{k0} = 1/p_k \tag{16}$$

a_{k0} : The total number of odd numbers N which are not divisible by any prime number of $(3, 5, 7, \cdots\cdots, p_{k-2}, p_{k-1})$ in $1 \leq N \leq (2*3*5*7*\cdots\cdots*p_{k-1}*p_k - 1)$

a_{kk} : The total number of multiples of p_k in a_{k0}

1.1.6 $m'(p_j)$ has 2 composite pairs of $(1, p_j^2)$ and $(p_j^2, 1)$ at $n = p_j^2 + 1$. Then we have the following (17).

$$0 < 2 \leq m'(p_j) \qquad (p_j^2 + 1 \leq n) \tag{17}$$

We have the following (18) from (17).

$$0 < 2/n \leq m'(p_j)/n \qquad (p_j^2 + 1 \leq n) \tag{18}$$

1.2 We have the following (19) from (7).

$$\begin{aligned} M'(n)/n =& m'(3)/n + m'(5)/n + m'(7)/n + \cdots + m'(p_k)/n + \cdots \\ &+ m'(p_{j-1})/n + m'(p_j)/n + (2,0)_n/n \\ &(k = 1, 2, 3, \cdots\cdots, j-1, j) \end{aligned} \tag{19}$$

1.2.1 $m(p_k)$ depends on whether n is divisible by p_k or not as shown in item 2.1 in the text. Since we calculated $m'(p_k)$ assuming that n is not divisible by any prime number of (3, 5, 7,$\cdots\cdots$, p_{j-1}, p_j), $m'(p_k)/n$ does not depend on n but only on k as shown in (12)—(16).

1.2.2 We have the following (20) from (16) and (18).

$$\begin{aligned} &m'(3)/n > m'(5)/n > m'(7)/n > \cdots > m'(p_k)/n > \\ &\cdots > m'(p_{j-1})/n > m'(p_j)/n > 0 \end{aligned} \tag{20}$$

We have the following (21) from (16) and (18). And we have (22) from (21) and $\lim_{j\to\infty} 1/p_j = 0$.

$$\begin{aligned} 0 < m'(p_j)/n \fallingdotseq\, & 2 * \{(1/2) - m'(3)/n - m'(5)/n - m'(7)/n - \cdots\cdots \\ & - m'(p_{j-2})/n - m'(p_{j-1})/n\}/p_j < 2 * (1/2)/p_j = 1/p_j \end{aligned} \tag{21}$$

$$\lim_{j\to\infty} m'(p_j)/n = 0 \tag{22}$$

1.2.3 $M'(n)/n$ does not depend on n but only on j from item 1.2.1. $M'(n)/n$ is a monotonically increasing function regarding j because $M'(n)/n$ has $(j+1)$ terms from (19) and if n is large enough, $(2,0)_n$ can be ignored in relation to $M'(n)$. Then if n is large enough i.e. if $4 * 10^{18} \leq n$, $\{1/2 - M'(n)/n\}$ decreases monotonically with increase of j. On the other hand $l'(n) = n * \{1/2 - M'(n)/n\}$ diverges to ∞ with $n \to \infty$ as shown in the following item 1.4.
If $\{1/2 - M'(n)/n \leq 0\}$ holds true in $j_0 \leq j$, $l'(n) \leq 0$ holds true in $(p_{j_0}^2 + 1) \leq n$ and $l'(n)$ cannot diverge to ∞ with $n \to \infty$. Therefore we have the following (23).

$$0 < 1/2 - M'(n)/n \qquad (4 * 10^{18} \leq n) \tag{23}$$

1.2.4 We can have the following (24) from (11) and (23).

$$0 < n * \{1/2 - M'(n)/n\} = l'(n) \leq l(n) \qquad (4 * 10^{18} \leq n) \tag{24}$$

We can have the following (25) from (24). Because $l(n)$ is the function that counts prime pairs, so $l(n)$ can have only integer values.

$$1 \leq l(n) \qquad (4 * 10^{18} \leq n) \tag{25}$$

1.3 We have the following (36) in [Appendix 2 : Investigation of $l(n)$].

$$l(n) \sim \frac{\{\pi(n)\}^2}{n/2} \sim \frac{\{n/\log n\}^2}{n/2} = \frac{2n}{(\log n)^2} \qquad (n \to \infty) \qquad (36)$$

We have the following (26) from (10), (11), (23) and the above (36).

$$0 < 1/2 - M'(n)/n = l'(n)/n \le l(n)/n \sim \frac{2}{(\log n)^2} \qquad (n \to \infty) \qquad (26)$$

We can have the following (27) from (26). We have the following (28) from (26) and (27).

$$\lim_{n\to\infty} l(n)/n = \lim_{n\to\infty} \frac{2}{(\log n)^2} = 0 \qquad (27)$$

$$\lim_{n\to\infty} \{1/2 - M'(n)/n\} = \lim_{j\to\infty} \{1/2 - M'(n)/n\} = 0 \qquad (28)$$

Memo 3

Since $M'(n)/n$ increases with increase of j, $\{1/2 - M'(n)/n\}$ decreases with increase of j. Then $l'(n) = n * \{1/2 - M'(n)/n\}$ decreases at $n = p_j^2 + 1$ because j increases by 1 at $n = p_j^2 + 1$. $l'(n)$ increases at the constant rate of $\{1/2 - M'(n)/n\}$ in $p_j^2 + 3 \le n \le p_{j+1}^2 - 1$. The constant rate of $\{1/2 - M'(n)/n\}$ converges to zero with $n \to \infty$ as shown in the above (28).

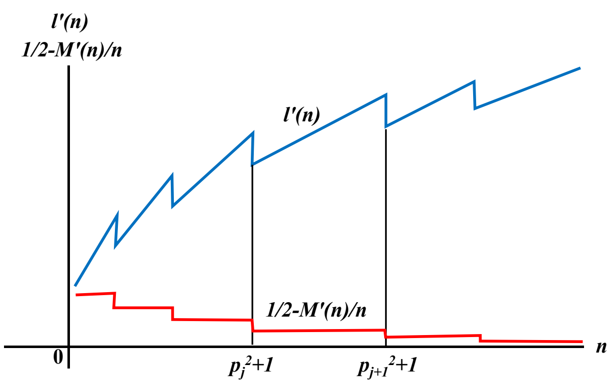


Figure 3 : $l'(n)$ and $\{1/2 - M'(n)/n\}$

1.4 $l'(n)$ diverges to ∞ with $n \to \infty$ as follows.
We have the following (29) from item 2.3 in the text because n is not divisible by any prime number of $(3, 5, 7, \cdots\cdots, p_{j-1}, p_j)$.

$$l(n) = l'(n)$$

$$(n = 2^b \quad b = 3, 4, 5, \cdots\cdots \quad \text{or} \quad n = 2 * p_c \quad c = 1, 2, 3, \cdots\cdots) \qquad (29)$$

We have the following (30) from (29).

$$l(2*p_c) = l'(2*p_c) \qquad (c = 1, 2, 3, \cdots\cdots) \tag{30}$$

We find the following in [Appendix 2 : Investigation of $l(n)$].

1.4.1 $l(n)$ is an almost increasing function regarding n.

1.4.2 $l(n)$ diverges to ∞ with $n \to \infty$.

We have the following (31) from (30) and item 1.4.2.

$$\lim_{n\to\infty} l'(n) = \lim_{c\to\infty} l'(2*p_c) = \lim_{c\to\infty} l(2*p_c) = \lim_{n\to\infty} l(n) = \infty \tag{31}$$

Memo 4

If $l'(n)$ converges to a certain positive value with $n \to \infty$, (30) cannot hold true in $n_0 < n$ from item 1.4.2. In other words, there exist only finite n that satisfy $n = 2*p_c$. This is a contradiction because there are infinite number of prime numbers. Therefore $l'(n)$ also diverges to ∞ with $n \to \infty$.

Appendix 2. : Investigation of $l(n)$

2.1 When an even number n is divided into 2 odd numbers x and y, we can find the pair of $\pi(n), l(n), m_{xx}, m_x, m_y$ and m_{xy} in $n/2$ pairs of (x, y) as shown in the following (Figure 4).

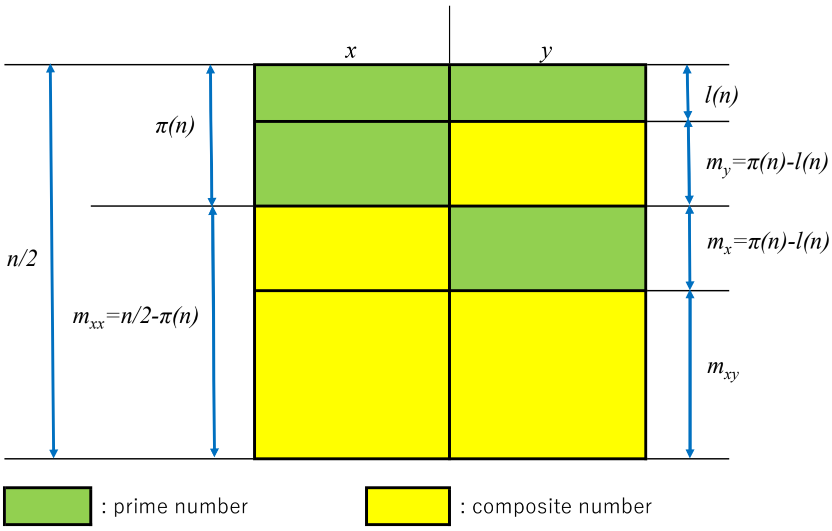


Figure 4 : Various pairs in $n/2$ pairs of (x, y)

We define as follows.

$\pi(n)$: $\pi(n)$ shows the total number of prime numbers which exist between 1 and n. But we use $\pi(n)$ in the above (Figure 4) for the total number of prime numbers which exist in $n/2$ odd numbers of $(1, 3, 5, \cdots\cdots, n-5, n-3, n-1)$. Strictly speaking, this value must be $\pi(n-1)-1$. But we can say $\pi(n-1)-1 = \pi(n)-1 \fallingdotseq \pi(n)$ because n is an even number and a large number as shown in (4).

m_{xx} : the total number of pairs where x is a composite number. 1 is regarded as a composite number.

m_x : the total number of pairs where x and y are composite number and prime number respectively

2.2 We have the following (32) from Prime number theorem.

$$\frac{\pi(n)}{n} \sim \frac{n/\log n}{n} = \frac{1}{\log n} \qquad (n \to \infty) \tag{32}$$

We have $\lim_{n\to\infty}\frac{\pi(n)}{n} = \lim_{n\to\infty}\frac{1}{\log n} = 0$ from the above (32). Then we have the following (33) from (Figure 4) and $\lim_{n\to\infty}\frac{\pi(n)}{n} = 0$

$$m_{xx} = n/2 - \pi(n) = (n/2)\{1 - 2\pi(n)/n\} \sim n/2 \qquad (n \to \infty) \tag{33}$$

When m_{xx} approaches $n/2$ with $n \to \infty$ as shown in the above (33), m_x approaches $\pi(n)$ with $n \to \infty$ due to the following reasons.

2.2.1 m_x shows the total number of prime numbers which exist in y of m_{xx} as shown in (Figure 4).

2.2.2 y of m_{xx} approaches $n/2$ odd numbers of $(1, 3, 5, \cdots\cdots, n-5, n-3, n-1)$ with $n \to \infty$ as shown in the above (33).

2.2.3 $(1, 3, 5, \cdots\cdots, n-5, n-3, n-1)$ has $\pi(n)$ prime numbers.

Then we can have the following (34) from (Figure 4).

$$m_x = \pi(n) - l(n) = \pi(n)\{1 - l(n)/\pi(n)\} \sim \pi(n) \qquad (n \to \infty) \tag{34}$$

We have $\lim_{n\to\infty}\frac{l(n)}{\pi(n)} = 0$ from the above (34). We have the following (35) from the above (33) and (34).

$$\frac{\pi(n) - l(n)}{n/2 - \pi(n)} \sim \frac{\pi(n)}{n/2} \qquad (n \to \infty) \tag{35}$$

We have the following (36) from the above (35) and Prime number theorem.

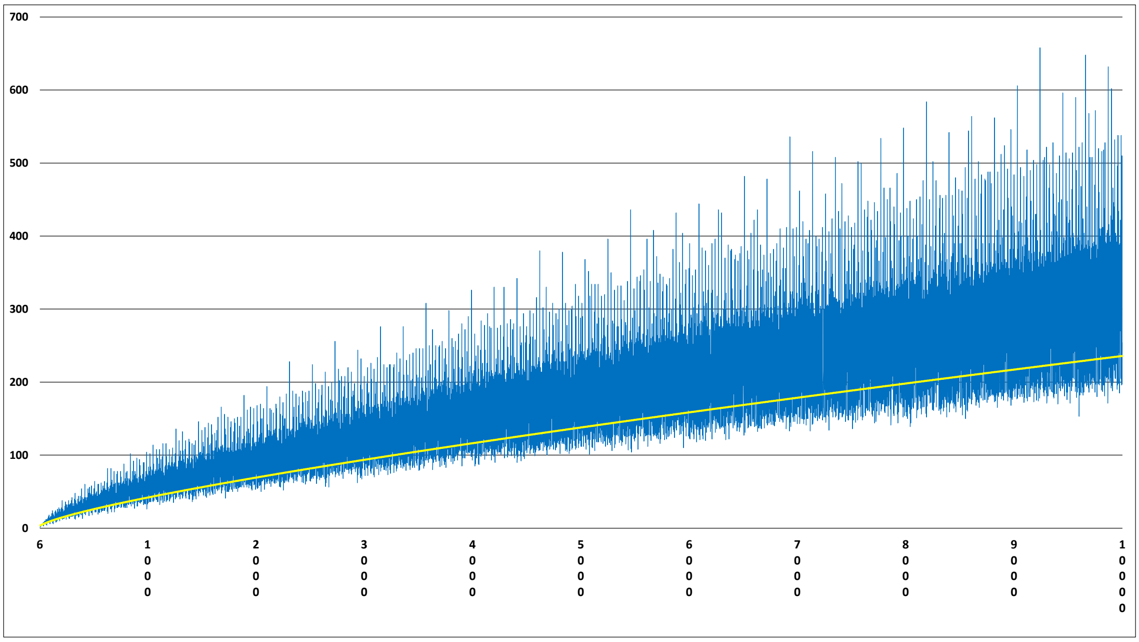
$$l(n) \sim \frac{\{\pi(n)\}^2}{n/2} \sim \frac{\{n/\log n\}^2}{n/2} = \frac{2n}{(\log n)^2} \qquad (n \to \infty) \tag{36}$$

We can find that $l(n)$ has the following properties from the above (36).

2.2.4 $l(n)$ repeats increases and decreases with increase of n as shown in the following (Graph 1). But overall $l(n)$ is an increasing function regarding n because $\frac{2n}{(\log n)^2}$ is an increasing function regarding n.

2.2.5 $l(n)$ diverges to ∞ with $n \to \infty$ because $\frac{2n}{(\log n)^2}$ diverges to ∞ with $n \to \infty$.

2.3 $\frac{2n}{(\log n)^2}$ seems to approximate $l(n)$ sufficiently well as shown in the following (Graph 1).



Graph 1 : $l(n)$(blue line)[1] and $\frac{2n}{(\log n)^2}$(yellow line) from $n = 6$ to $n = 10,000$

References

[1] THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES

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