Proof of Goldbach conjecture

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Abstract. This paper is a trial to prove Goldbach conjecture according to the following process.

1. We make the fictitious function $l'(n)$ regarding n. When an even number n is divided into 2 odd numbers, $l'(n) \leq l(n)$ holds true.
   $l(n)$: the total number of ways to divide an even number n into 2 prime numbers.
2. We find that $1 < l'(n)$ holds true in $122 \leq n$.
3. Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$.
4. Goldbach conjecture is true from the above item 1. — 3.

1. Introduction

1.1 When an even number n is divided into 2 odd numbers x and y, we can express the situation as pair (x, y) like the following (1).

$$n = x + y = (x, y) \qquad (n = 6, 8, 10, 12, \cdots\cdots \quad x, y : \text{odd number}) \tag{1}$$

n has $n/2$ pairs like the following (2).

$$(1, n-1), (3, n-3), (5, n-5), \cdots\cdots, (n-5, 5), (n-3, 3), (n-1, 1) \tag{2}$$

We define as follows.

Prime pair : the pair where both x and y in (x, y) are prime numbers
Composite pair : the pair other than the above prime pair
$l(n)$: the total number of the prime pairs which exist in $n/2$ pairs shown by the above (2). (p, q) is regarded as the different pair from (q, p). (p, q : prime number)

1.2 Goldbach conjecture can be expressed as the following (3).

$$1 \leq l(n) \qquad (n = 6, 8, 10, 12, \cdots\cdots) \tag{3}$$

Goldbach conjecture is confirmed to be true up to $n = 4 * 10^{18}$. So we will try to prove Goldbach conjecture in the following condition.

$$4 * 10^{18} < n \tag{4}$$

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2. The total number of composite pair in $n/2$ pairs

2.1 We can calculate the total number of composite pair where x or y is divisible by p in $n/2$ pairs as follows. (p : prime number)

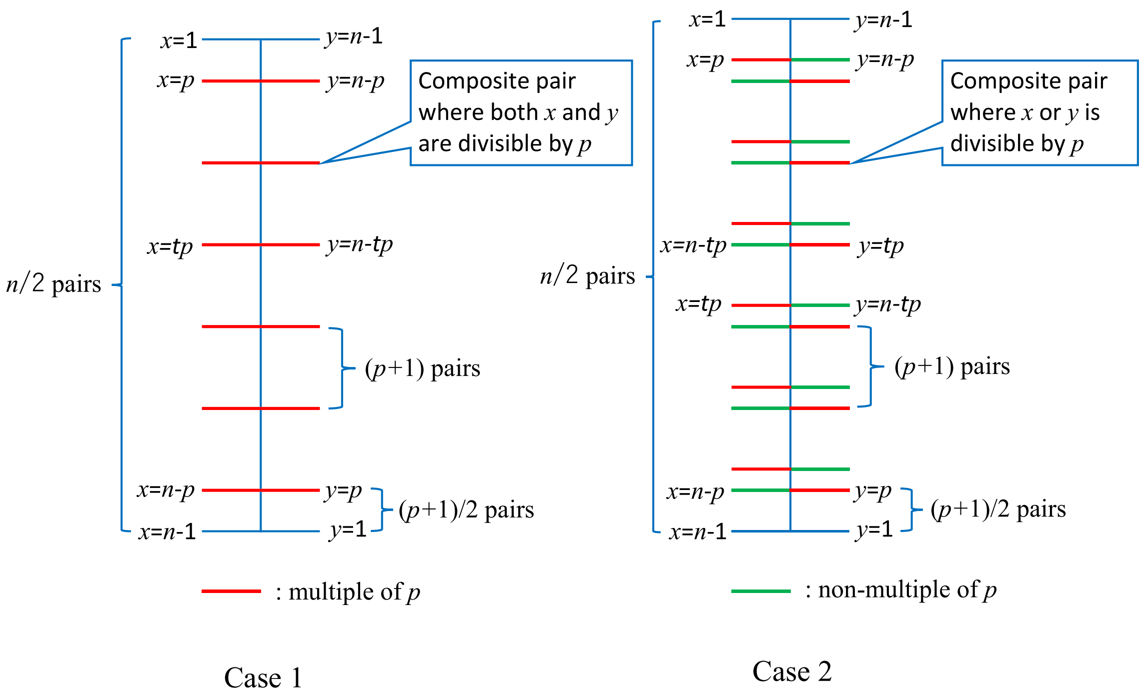


Figure 1 : Case 1 and Case 2

Case 1 : When n is divisible by p, both x and y are divisible by p in the pair of $(x, y) = (tp, n - tp)$. $(t = 1, 3, 5, 7, \cdots\cdots, n/p - 5, n/p - 3, n/p - 1)$ Then $(n/p)/2$ composite pairs where both x and y are divisible by p exist in $n/2$ pairs.

Case 2 : When n is not divisible by p, $(n - tp)$ is not divisible by p in the pair of $(x, y) = (tp, n - tp)$. $(t = 1, 3, 5, 7, \cdots\cdots, 2 * \lfloor (n/2)/p \rfloor - 5, 2 * \lfloor (n/2)/p \rfloor - 3, 2 * \lfloor (n/2)/p \rfloor - 1)$ Then $\lfloor (n/2)/p \rfloor$ composite pairs where x is divisible by p exist in $n/2$ pairs as shown in the following [Memo 1]. If $(n - p)$ is a prime number at $t = 1$, $(p, n - p)$ is a prime pair and $(\lfloor (n/2)/p \rfloor - 1)$ composite pairs exist. Since the case of $(x, y) = (n - tp, tp)$ is also similar, the total number of composite pairs where x or y is divisible by p is $2 * \lfloor (n/2)/p \rfloor$ or $(2 * \lfloor (n/2)/p \rfloor - 2)$. $2 * \lfloor (n/2)/p \rfloor$ and $(2 * \lfloor (n/2)/p \rfloor - 2)$ can be approximated to $\lfloor n/p \rfloor$ because n is a large number as shown in (4).

Memo 1

If we divide $n/2$ odd numbers of $(1,\ 3,\ 5,\ \cdots\cdots, n-1)$ into groups of p, there will always be a multiple of p in the middle of each group as shown in the following (Figure 2). When $0 \leq \text{frac}\{(n/2)/p\} < 0.5$ holds true as shown in Case 3, the total number of multiples of p is $\lfloor (n/2)/p \rfloor$. Frac(x) means $(x - \lfloor x \rfloor)$. When $0.5 \leq \text{frac}\{(n/2)/p\} < 1$ holds true as shown in Case 4, the total number of multiples of p is $\lceil (n/2)/p \rceil$. Then we can say that the total number of multiples of p is $\lfloor (n/2)/p + 0.5 \rfloor$. $\lfloor (n/2)/p + 0.5 \rfloor$ can be approximated to $\lfloor (n/2)/p \rfloor$ because n is a large number as shown in (4).

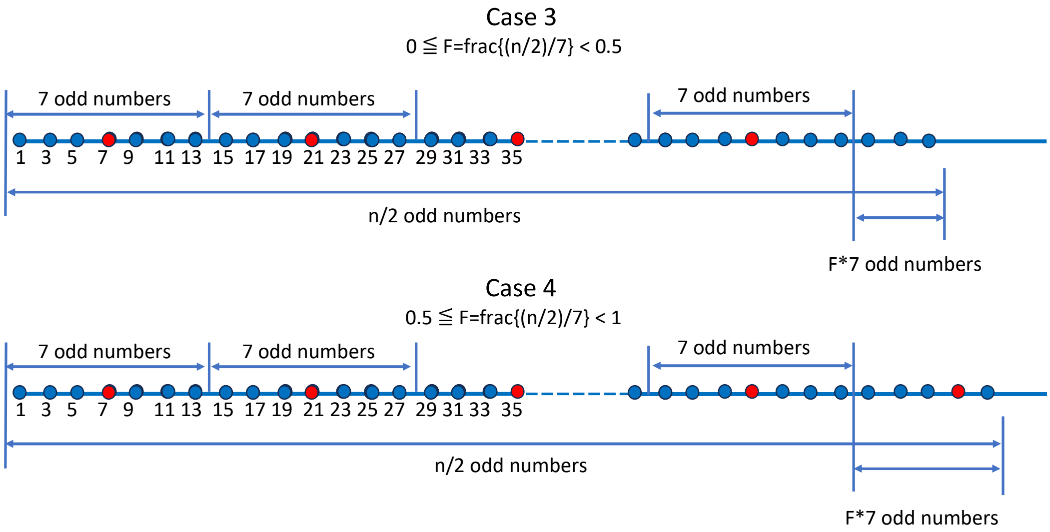


Figure 2 ： Case 3 and Case 4

2.2 We express prime numbers as follows.

$p_1 = 3,\ p_2 = 5,\ p_3 = 7, \cdots\cdots$

$(3,\ 5,\ 7, \cdots\cdots, p_{j-1}, p_j)$ are prime numbers arranged in ascending order.

From the above item 2.1 we can calculate $M(n)$: {the total number of composite pair in $n/2$ pairs} as the following (5).

$$M(n) = m(3) + m(5) + m(7) + \cdots + m(p_k) + \cdots + m(p_{j-1}) + m(p_j) + (2,0)_n \quad (k = 1, 2, 3, \cdots\cdots, j-1, j) \tag{5}$$

$m(p_k)$: the total number of pairs where x or y is a composite number and divisible by p_k but not divisible by any prime number of $(3,\ 5,\ 7, \cdots\cdots, p_{k-1})$

p_j : the largest prime number that satisfies $p < \sqrt{n}$.

$(2,0)_n$: when $(n-1)$ is a prime number, $(2,0)_n = 2$ holds true because both $(1, n-1)$ and $(n-1, 1)$ are composite pairs. When $(n-1)$ is a composite number, $(2,0)_n = 0$ holds true. If n is large enough, $(2,0)_n$ can be ignored.

We have the following (6) from the above (5).

$$l(n) = n/2 - M(n) \tag{6}$$

2.3 Here we define an imaginary function $M'(n)$ like the following (7).

$$M'(n) = m'(3) + m'(5) + m'(7) + \cdots + m'(p_k) + \cdots + m'(p_{j-1}) + m'(p_j) + (2,0)_n \tag{7}$$

$m'(p_k)$: the total number of pairs where x or y is a composite number and divisible by p_k but not divisible by any prime number of (3, 5, 7, $\cdots\cdots, p_{k-1}$) when we assume that n is not divisible by p_k

We can have the above $M'(n)$ by assuming that n is not divisible by any prime number of (3, 5, 7,$\cdots\cdots, p_{j-1}, p_j$) like $n = 2^b$ or $n = 2*p_c$. $(b = 3, 4, 5, \cdots\cdots \quad c = 1, 2, 3, \cdots\cdots)$ In other words we calculate $m'(p_k)$ by doubling the total number of x which is a composite number and divisible by p_k but not divisible by any prime number of (3, 5, 7,$\cdots\cdots, p_{k-1}$).

We can have the following (8) from item 2.1 and the above difinition of $m(p_k)$ and $m'(p_k)$. We have the following (9) from (8).

$$m(p_k) \leq m'(p_k) \qquad (k = 1, 2, 3, \cdots\cdots, j) \tag{8}$$

$$M(n) \leq M'(n) \tag{9}$$

2.4 We define $l'(n)$ as the following (10) and we have the following (11) from (6), (9) and (10).

$$l'(n) = n/2 - M'(n) \tag{10}$$

$$l'(n) = n/2 - M'(n) \leq n/2 - M(n) = l(n) \tag{11}$$

3. The property of $l'(n)$

3.1 We can calculate $m'(p_k)$ on the condition that n is not divisible by p_k as follows.

3.1.1 $m'(3)$ can be calculated as the following (12).

$$m'(3) = 2 * \lfloor (n/2)/3 + 0.5 \rfloor - 2 \fallingdotseq 2 * (n/2)/3 = 0.33 * n \tag{12}$$

3.1.2 $m'(5)$ can be calculated as the following (13).

$$m'(5) = 2 * \lfloor \{(n/2) - m'(3)\} * a_2 + 0.5 \rfloor - 2 \fallingdotseq 0.068 * n$$

$$a_2 = \frac{(3*5)\{1/5 - 1/(3*5)\}}{(3*5)(1 - 1/3)} = 1/5 \tag{13}$$

$\{(n/2) - m'(3)\}$ is the total number of pairs where both x and y are not divisible by 3. a_2 is the proportion of (the total number of multiples of 5) in the total number of odd numbers N which are not divisible by 3 in $1 \leq N \leq (2*3*5-1)$.

Memo 2

5 is a multiple of 5 but not a composite number. a_2 is calculated including 5. Then we must subtract 2 in the above (13). The same applies to the above (12) and the following (14)—(16).

3.1.3 $m'(7)$ can be calculated as the following (14).

$$\begin{aligned} m'(7) =& 2 * \lfloor \{(n/2) - m'(3) - m'(5)\} * a_3 + 0.5 \rfloor - 2 \fallingdotseq 0.029 * n \\ a_3 =& \frac{(3*5*7)\{1/7 - 1/(3*7) - 1/(5*7) + 1/(3*5*7)\}}{(3*5*7)\{1 - 1/3 - 1/5 + 1/(3*5)\}} = 1/7 \end{aligned} \tag{14}$$

$\{(n/2) - m'(3) - m'(5)\}$ is the total number of pairs where both x and y are not divisible by either 3 or 5. a_3 is the proportion of (the total number of multiples of 7) in the total number of odd numbers N which are not divisible by 3 or 5 in $1 \leq N \leq (2*3*5*7-1)$.

3.1.4 Similarly $m'(11)$ can be calculated as the following (15).

$$m'(11) = 2 * \lfloor \{(n/2) - m'(3) - m'(5) - m'(7)\} * a_4 + 0.5 \rfloor - 2 \fallingdotseq 0.013 * n \tag{15}$$

$$a_4 = \frac{(3*5*7*11)\{1/11-1/(3*11)-1/(5*11)-1/(7*11)+1/(3*5*11)+1/(5*7*11)+1/(7*3*11)-2/(3*5*7*11)\}}{(3*5*7*11)\{1-1/3-1/5-1/7+1/(3*5)+1/(5*7)+1/(7*3)-2/(3*5*7)\}} = 1/11$$

Memo 3

In odd numbers N which are not divisible by 3, 5, or 7 in $1 \leq N \leq (2*3*5*7*11-1)$ and are arranged in ascending order, the multiples of 11 occur at intervals of 11 on average, although there is some variation. As mentioned in [Memo 1], $\lfloor x+0.5 \rfloor$ is more accurate than $\lfloor x \rfloor$ or $\lceil x \rceil$ when calculating the total number of the multiples of p that are approximately equally spaced among the odd numbers and converting the calculation result x to an integer. The same applies to the above (12)—(14) and the following (16).

3.1.5 $m'(p_j)$ can be calculated as the following (16).

$$\begin{aligned} & m'(p_j) = \\ & 2 * \lfloor \{(n/2) - m'(3) - m'(5) - m'(7) - \cdots\cdots - m'(p_{j-2}) - m'(p_{j-1})\} * a_j + 0.5 \rfloor - 2 \\ & \qquad a_j = a_{jj}/a_{j0} = 1/p_j \end{aligned} \tag{16}$$

a_{j0} : The total number of odd numbers N which are not divisible by any prime number of $(3, 5, 7, \cdots\cdots, p_{j-2}, p_{j-1})$ in $1 \leq N \leq (2*3*5*7*\cdots\cdots*p_{j-1}*p_j-1)$

a_{jj} : The total number of multiples of p_j in a_{j0}

3.2 $m'(p_j)$ has 2 composite pairs of $(1, p_j^2)$ and $(p_j^2, 1)$ at $n = p_j^2 + 1$. Then we have the following (17).

$$2 \leq m'(p_j) \qquad\qquad (p_j^2 + 1 \leq n) \tag{17}$$

3.3 We have the following (18) from (7) and (16). $\leq$ in (18) holds true from $\lfloor x \rfloor \leq x$.

$$\begin{aligned} & m'(p_j) = \\ & 2 * \lfloor \{(n/2) - m'(3) - m'(5) - m'(7) - \cdots\cdots - m'(p_{j-2}) - m'(p_{j-1})\}/p_j + 0.5 \rfloor - 2 \\ & < 2 * \lfloor \{(n/2) - m'(3) - m'(5) - m'(7) - \cdots\cdots - m'(p_{j-2}) - m'(p_{j-1})\}/p_j + 0.5 \rfloor \\ & = 2 * \lfloor \{n/2 - M'(n) + m'(p_j) + (2,0)_n\}/p_j + 0.5 \rfloor \\ & \leq 2 * \{n/2 - M'(n) + m'(p_j) + (2,0)_n\}/p_j + 1 \end{aligned} \tag{18}$$

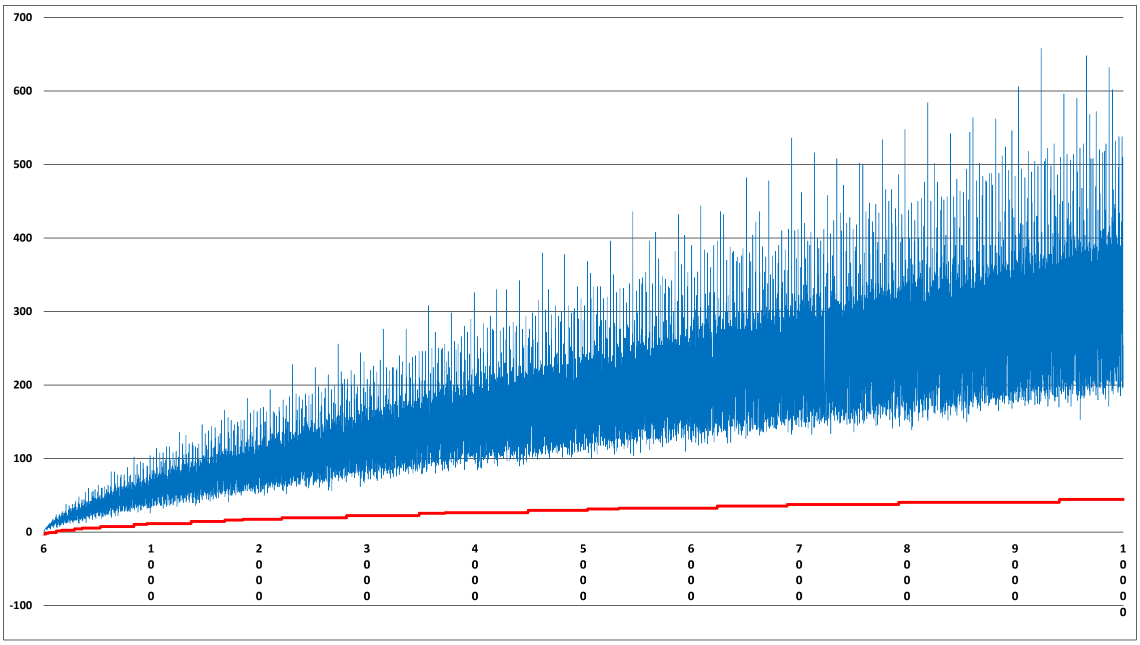
We have the following (19) from (10), (11), (17) and (18).

$$\begin{aligned} l(n) \geq & l'(n) = n/2 - M'(n) > (p_j/2 - 1) * m'(p_j) - p_j/2 - (2,0)_n \\ \geq & p_j/2 - 2 - (2,0)_n \geq p_j/2 - 4 \end{aligned} \tag{19}$$

We have the following (20) from (19).

$$\begin{aligned} & 1 < p_j/2 - 4 < l'(n) \leq l(n) \qquad\qquad (4 \leq j \quad p_4^2 + 1 = 122 \leq n) \\ & \qquad p_j : \text{the largest prime number that satisfies } p < \sqrt{n} \end{aligned} \tag{20}$$

The following (Graph 1) shows $l(n)$ and $(p_j/2 - 4)$.



Graph 1 : $l(n)$(blue line)[1] and $\{p_j/2 - 4\}$(red line) from $n = 6$ to $n = 10,000$

3.4 We can find that $l(n)$ has the following properties from the above (20).

3.4.1 We have the following (21).

$$1 < l(n) \qquad\qquad (122 \leq n) \tag{21}$$

3.4.2 $l(n)$ diverges to ∞ with $n \to \infty$ because p_j diverges to ∞ with $j \to \infty$ i.e. $n \to \infty$.

4. Conclusion

Goldbach conjecture is true from the following item 4.1 and 4.2.

4.1 Goldbach conjecture is true in $122 \leq n$ from (3) and the above (21).

4.2 Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$.

References

[1] THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES

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