Proof of Goldbach conjecture

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Abstract. This paper is a trial to prove Goldbach conjecture according to the following process.

1. When an even number n is divided into 2 odd numbers, we make $l(n)$ which shows the total number of ways to divide an even number n into 2 prime numbers.
2. We find that $1 < l(n)$ holds true in $122 \leq n$.
3. Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$.
4. Goldbach conjecture is true from the above item 2 and 3.

1. Introduction

1.1 When an even number n is divided into 2 odd numbers x and y, we can express the situation as pair (x, y) like the following (1).

$$n = x + y = (x, y) \qquad (n = 6, 8, 10, 12, \cdots\cdots \quad x, y : \text{odd number}) \tag{1}$$

n has $n/2$ pairs like the following (2).

$$(1, n-1), (3, n-3), (5, n-5), \cdots\cdots, (n-5, 5), (n-3, 3), (n-1, 1) \tag{2}$$

We define as follows.

Prime pair : the pair where both x and y in (x, y) are prime numbers

Composite pair : the pair other than the above prime pair. 1 is regarded as a composite number.

$l(n)$: the total number of the prime pairs which exist in $n/2$ pairs shown by the above (2). (p, q) is regarded as the different pair from (q, p). $(p, q$: prime number$)$

1.2 Goldbach conjecture can be expressed as the following (3).

$$1 \leq l(n) \qquad (n = 6, 8, 10, 12, \cdots\cdots) \tag{3}$$

Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$. So we can try to prove Goldbach conjecture in the following condition.

$$4 * 10^{18} < n \tag{4}$$

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2. The total number of composite pair in $n/2$ pairs

2.1 We can calculate the total number of composite pair where x or y is a composite number and divisible by p in $n/2$ pairs as follows. (p : prime number)

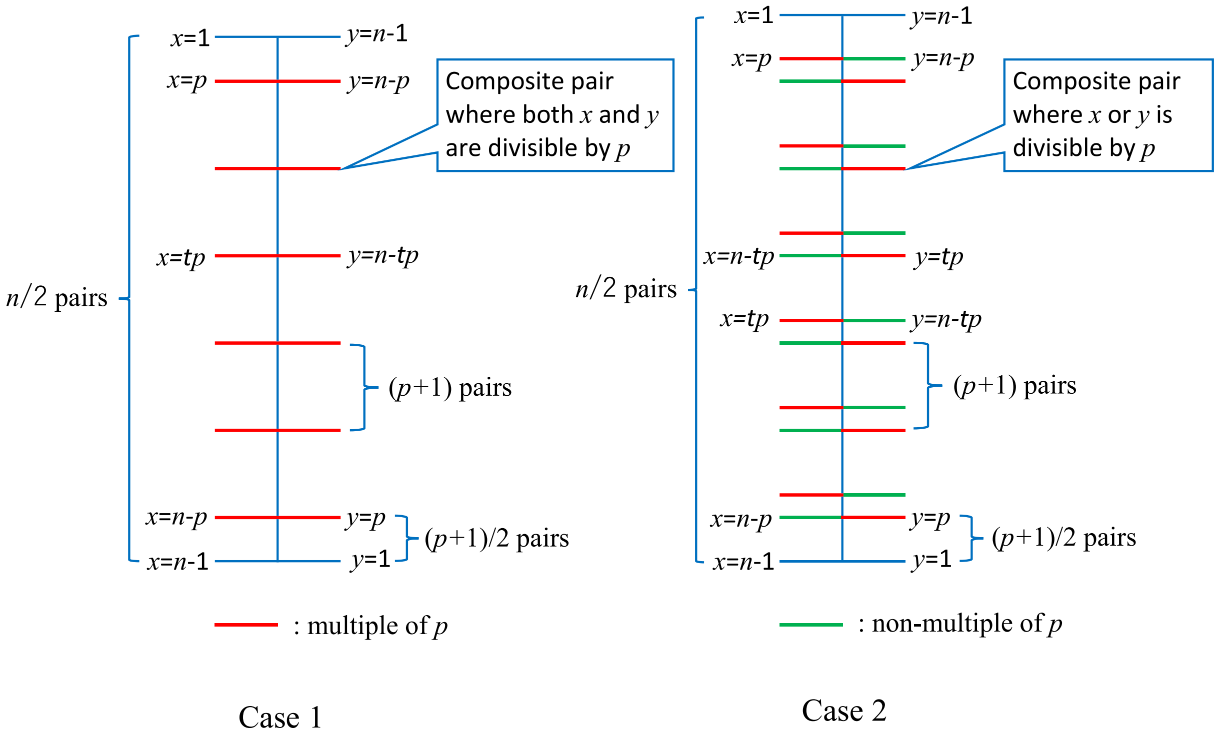


Figure 1 : Case 1 and Case 2

Case 1 : When n is divisible by p, both x and y are divisible by p in the pair of $(x, y) = (tp, n - tp)$. $(t = 1, 3, 5, 7, \cdots\cdots, n/p - 5, n/p - 3, n/p - 1)$ Then $(n/p)/2$ composite pairs where both x and y are divisible by p exist in $n/2$ pairs.

Case 2 : When n is not divisible by p, $(n - tp)$ is not divisible by p in the pair of $(x, y) = (tp, n - tp)$. $(t = 1, 3, 5, 7, \cdots\cdots, 2 * \lfloor (n/2)/p + 0.5 \rfloor - 5, 2 * \lfloor (n/2)/p + 0.5 \rfloor - 3, 2 * \lfloor (n/2)/p + 0.5 \rfloor - 1)$ Then $\lfloor (n/2)/p + 0.5 \rfloor$ pairs where x is divisible by p exist in $n/2$ pairs as shown in the following [Memo 1]. Since p is not a composite number, $(p, n-p)$ is not a composite pair where x is a composite number and divisible by p. Then $(\lfloor (n/2)/p + 0.5 \rfloor - 1)$ composite pairs where x is a composite number and divisible by p exist. Since the case of $(x, y) = (n - tp, tp)$ is also similar, the total number of composite pairs where x or y is a composite number and divisible by p is $(2 * \lfloor (n/2)/p + 0.5 \rfloor - 2)$.

Memo 1

If we divide $n/2$ odd numbers of $(1,\ 3,\ 5,\ \cdots\cdots, n-3, n-1)$ into groups of p, there will always be a multiple of p in the middle of each group as shown in the following (Figure 2). When $0 \leq \text{frac}\{(n/2)/p\} < 0.5$ holds true as shown in Case 3, the total number of multiples of p is $\lfloor (n/2)/p \rfloor$. Frac(x) means $(x - \lfloor x \rfloor)$. When $0.5 \leq \text{frac}\{(n/2)/p\} < 1$ holds true as shown in Case 4, the total number of multiples of p is $\lceil (n/2)/p \rceil$. Then we can say that the total number of multiples of p is $\lfloor (n/2)/p + 0.5 \rfloor$ in both Case 3 and Case 4.

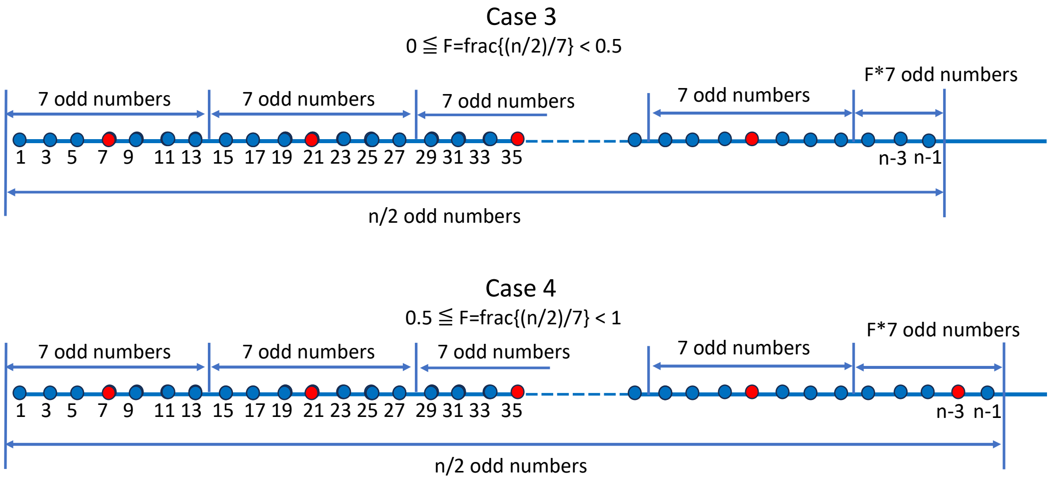


Figure 2 ： Case 3 and Case 4

2.2 We express prime numbers as follows.

$p_1 = 3,\ p_2 = 5,\ p_3 = 7, \cdots\cdots$

$(3,\ 5,\ 7, \cdots\cdots, p_{j-1}, p_j)$ are prime numbers arranged in ascending order.
From the above item 2.1 we can calculate $M(n)$: {the total number of composite pair in $n/2$ pairs} as the following (5).

$$M(n) = m(3) + m(5) + m(7) + \cdots + m(p_k) + \cdots + m(p_{j-1}) + m(p_j) + (0,2)_{n-1}$$
$$(k = 1, 2, 3, \cdots\cdots, j-1, j) \tag{5}$$

$m(p_k)$: the total number of pairs where x or y is a composite number and divisible by p_k but not divisible by any prime number of $(3,\ 5,\ 7, \cdots\cdots, p_{k-2}, p_{k-1})$

p_j : the largest prime number that satisfies $p < \sqrt{n}$

Memo 2

$m(p_j)$ has 2 composite pairs of $(1, p_j^2)$ and $(p_j^2, 1)$ at $n = p_j^2 + 1$. Then x or y that are composite numbers and divisible by p_j but not divisible by any prime number of $(3, 5, 7, \cdots\cdots, p_{j-2}, p_{j-1})$ in $(p_j^2 + 1 \leq n \leq p_{j+1}^2 - 1)$ are $(p_j^2, p_j * p_{j+1}, p_j * p_{j+2}, p_j * p_{j+3}, \cdots\cdots, p_j * p_{j+E})$. $(p_j * p_{j+E} < p_{j+1}^2)$ Therefore if we divide x or y by prime numbers from 3 to p_j, we can determine whether x or y is a composite number or not in $(p_j^2+1 \leq n \leq p_{j+1}^2 - 1)$. Since $(p_j < \sqrt{p_j^2 + 1} \leq \sqrt{n} \leq \sqrt{p_{j+1}^2 - 1} < p_{j+1})$ holds true, p_j is the largest prime number that satisfies $p < \sqrt{n}$.

$(0,2)_{n-1}$: $(0,2)_{n-1} = 0$ in $(n-1$: composite number$)$
$(0,2)_{n-1} = 2$ in $(n-1$: prime number$)$

$(1, n-1)$ and $(n-1, 1)$ are not included in any of $\{m(3), m(5), m(7), \cdots\cdots,$ $m(p_{j-1}), m(p_j)\}$ when $(n-1)$ is a prime number.
We have the following (6) from the above (5).

$$l(n) = n/2 - M(n) \tag{6}$$

3. The property of $l(n)$

3.1 We can calculate $m(p_k)$ from item 2.1 as follows.

3.1.1 $m(3)$ can be calculated as the following (7).

$$m(3) = (1,2)_{3|n} * \lfloor (n/2)/3 + 0.5 \rfloor - (0,2)_{3|n} \tag{7}$$

$(1,2)_{3|n}$: $(1,2)_{3|n} = 1$ in $3|n$ $\quad (1,2)_{3|n} = 2$ in $3 \nmid n$
$(0,2)_{3|n}$: $(0,2)_{3|n} = 0$ in $3|n$ $\quad (0,2)_{3|n} = 2$ in $3 \nmid n$
$(1,2)_{p_k|n}$ is defined in the same way as above.

Memo 3

3 is a multiple of 3 but not a composite number. $\lfloor (n/2)/3 + 0.5 \rfloor$ is the total number of multiples of 3 in x or y as shown in [Memo 1]. We cannot include $(3, n-3)$ and $(n-3, 3)$ into $m(3)$ when n is not divisible by 3 as shown in Case 2 in item 2.1. Then we must subtract $(0,2)_{3|n} = 2$ in the above (7). The same applies to the following (8),(10), (12) and (13).

3.1.2 $m(5)$ can be calculated as the following (8).

$$m(5) = (1,2)_{5|n} * \lfloor \{n/2 - m(3)\} * a_2 + 0.5 \rfloor - (0,2)_{5|n,3|(n-5)} \tag{8}$$

$$a_2 = \frac{(n/2)\{1/5 - (1,2)_{3|n}/(3*5)\}}{(n/2)\{1 - (1,2)_{3|n}/3\} + (0,2)_{3|n}} \fallingdotseq \frac{(n/2)\{1/5 - (1,2)_{3|n}/(3*5)\}}{(n/2)\{1 - (1,2)_{3|n}/3\}} = 1/5 \tag{9}$$

$(0,2)_{5|n,3|(n-5)}$: $(0,2)_{5|n,3|(n-5)} = 0$ in $\{5|n\}$ or $\{5 \nmid n$ and $3|(n-5)\}$
$(0,2)_{5|n,3|(n-5)} = 2$ in $\{5 \nmid n$ and $3 \nmid (n-5)\}$

$\{n/2 - m(3)\}$ is the total number of the following 3 types of pairs.

(1) Both x and y are composite numbers not divisible by 3.

(2) One of x and y is a composite number not divisible by 3, and the other is a prime number.

(3) Both x and y are prime numbers.

a_2 is the proportion of (the total number of multiples of 5) in the total number of x or y of $\{n/2 - m(3)\}$ pairs.

Memo 4

Whether $(5, n-5)$ and $(n-5, 5)$ are included in $m(5)$ depends on whether n is divisible by 5 or not as shown in item 2.1. However, if $(n-5)$ is divisible by 3, then $(5, n-5)$ and $(n-5, 5)$ are already included in $m(3)$ and they cannot exist in $m(5)$. Therefore $(0,2)_{5|n,3|(n-5)}$ is defined as above.
When n is not divisible by 3, $\{n/2 - m(3)\}$ pairs include $(3, n-3)$ and $(n-3, 3)$ from (7). But $(n/2)\{1-(1,2)_{3|n}/3\}$ pairs do not include these 2 pairs. Then we must add $(0,2)_{3|n}$ to $(n/2)\{1-(1,2)_{3|n}/3\}$ as shown in the above (9).
Ignoring $(0,2)_{3|n}$ in (9) overestimates m(5) and it underestimates $l(n)$ from (5) and (6). So ignoring $(0,2)_{3|n}$ does not affect the proof of (3).

3.1.3 $m(7)$ can be calculated as the following (10).

$$m(7) = (1,2)_{7|n} * \lfloor \{n/2 - m(3) - m(5)\} * a_3 + 0.5 \rfloor - (0,2)_{7|n,(3\sim5)|(n-7)} \quad (10)$$

$$a_3 = \frac{(n/2)\{1/7-(1,2)_{3|n}/(3*7)-(1,2)_{5|n}/(5*7)+(1,2)_{3|n}*(1,2)_{5|n}/(3*5*7)\}}{(n/2)\{1-(1,2)_{3|n}/3-(1,2)_{5|n}/5+(1,2)_{3|n}*(1,2)_{5|n}/(3*5)\}+\sum_{p_k=3}^{5}(0,2)_{p_k|n,(3\sim p_{k-1})|(n-p_k)}}$$

$$\fallingdotseq \frac{(n/2)\{1/7-(1,2)_{3|n}/(3*7)-(1,2)_{5|n}/(5*7)+(1,2)_{3|n}*(1,2)_{5|n}/(3*5*7)\}}{(n/2)\{1-(1,2)_{3|n}/3-(1,2)_{5|n}/5+(1,2)_{3|n}*(1,2)_{5|n}/(3*5)\}}$$

$$= 1/7 \quad (11)$$

We define as follows.

$(3\sim5)|(n-7)$: $(n-7)$ is divisible by at least one prime number between 3 and 5.

$(3\sim5)\nmid(n-7)$: $(n-7)$ is not divisible by any of the prime numbers from 3 to 5.

$(0,2)_{7|n,(3\sim5)|(n-7)}$: $(0,2)_{7|n,(3\sim5)|(n-7)} = 0$ in $\{7|n\}$ or $\{7\nmid n$ and $(3\sim5)|(n-7)\}$

$(0,2)_{7|n,(3\sim5)|(n-7)} = 2$ in $\{7\nmid n$ and $(3\sim5)\nmid(n-7)\}$

$(0,2)_{p_k|n,(3\sim p_{k-1})|(n-p_k)}$ is defined in the same way as above. We define $(0,2)_{p_k|n,(3\sim p_{k-1})|(n-p_k)} = (0,2)_{3|n}$ at $p_k = 3$.

$\{n/2 - m(3) - m(5)\}$ is the total number of the following 3 types of pairs.

(1) Both x and y are composite numbers not divisible by either 3 or 5.

(2) One of x and y is a composite number not divisible by either 3 or 5, and the other is a prime number.

(3) Both x and y are prime numbers.

a_3 is the proportion of (the total number of multiples of 7) in the total number of x or y of $\{n/2 - m(3) - m(5)\}$ pairs.
We must subtract $(0,2)_{7|n,(3\sim 5)|(n-7)}$ in (10) due to the same reason as in [Memo 4].
We must add $\sum_{p_k=3}^{5} (0,2)_{p_k|n,(3\sim p_{k-1})|(n-p_k)}$ in (11) due to the same reason as in [Memo 4].
Ignoring $\sum_{p_k=3}^{5} (0,2)_{p_k|n,(3\sim p_{k-1})|(n-p_k)}$ in (11) overestimates $m(7)$ and it underestimates $l(n)$ from (5) and (6). So it does not affect the proof of (3).

Memo 5

While the multiples of 7 occur at intervals of 7 among consecutive odd numbers, in x or y of $\{(n/2) - m(3) - m(5)\}$ pairs which are arranged in ascending order, the multiples of 7 also occur at intervals of 7 on average as a_3 shows. This is because, while multiples of 3 or 5 exist at intervals of 3 or 5 among consecutive odd numbers, multiples of 3 or 5 also exist at the same intervals of 3 or 5 among consecutive multiples of 7.
As mentioned in [Memo 1], $\lfloor x+0.5 \rfloor$ is more accurate than $\lfloor x \rfloor$ or $\lceil x \rceil$ when calculating the total number of the multiples of p that are approximately equally spaced among the odd numbers and converting the calculation result x to an integer. The same applies to the above (7), (8) and the following (12) and (13).

3.1.4 Similarly $m(11)$ can be calculated as the following (12).

$$m(11) = \\ (1,2)_{11|n} * \lfloor \{n/2 - m(3) - m(5) - m(7)\} * a_4 + 0.5 \rfloor - (0,2)_{11|n,(3\sim 7)|(n-11)} \tag{12}$$

$$a_4 = a_{4*4}/a_{4*0} \fallingdotseq 1/11$$

$$\begin{aligned} a_{4*0} = (n/2)\{&1 - (1,2)_{3|n}/3 - (1,2)_{5|n}/5 - (1,2)_{7|n}/7 + (1,2)_{3|n} * (1,2)_{5|n}/(3*5) \\ &+ (1,2)_{5|n} * (1,2)_{7|n}/(5*7) + (1,2)_{7|n} * (1,2)_{3|n}/(7*3) \\ &- 2*(1,2)_{3|n} * (1,2)_{5|n} * (1,2)_{7|n}/(3*5*7)\} + \sum_{p_k=3}^{7} (0,2)_{p_k|n,(3\sim p_{k-1})|(n-p_k)} \end{aligned}$$

$$\begin{aligned} a_{4*4} = (n/2)\{&1/11 - (1,2)_{3|n}/(3*11) - (1,2)_{5|n}/(5*11) - (1,2)_{7|n}/(7*11) \\ &+ (1,2)_{3|n} * (1,2)_{5|n}/(3*5*11) + (1,2)_{5|n} * (1,2)_{7|n}/(5*7*11) \\ &+ (1,2)_{7|n} * (1,2)_{3|n}/(7*3*11) - 2*(1,2)_{3|n} * (1,2)_{5|n} * (1,2)_{7|n}/(3*5*7*11)\} \\ \fallingdotseq (1/11)&* a_{4*0} \end{aligned}$$

3.1.5 $m(p_j)$ can be calculated as the following (13).

$$m(p_j) = (1,2)_{p_j|n} * \lfloor \{n/2 - m(3) - m(5) - m(7) - \cdots\cdots - m(p_{j-2})$$

$$- m(p_{j-1})\} * a_j + 0.5\rfloor - (0,2)_{p_j|n,(3\sim p_{j-1})|(n-p_j)} \tag{13}$$

$$a_j = a_{jj}/a_{j0} \fallingdotseq 1/p_j$$

a_{j0} : {The remainder after removing the pairs of (multiples of p_c, $n-$ multiples of p_c) and ($n-$ multiples of p_c, multiples of p_c) from $n/2$ pairs} plus $\sum_{p_k=3}^{p_{j-1}} (0,2)_{p_k|n,(3\sim p_{k-1})|(n-p_k)}$ $\quad (p_c = 3, 5, 7, \cdots\cdots, p_{j-2}, p_{j-1})$

a_{jj} : The total number of multiples of p_j in a_{j0}

3.2 x or y that are composite numbers and divisible by p_j but not divisible by any prime number of $(3, 5, 7, \cdots\cdots, p_{j-2}, p_{j-1})$ in $(p_j^2 + 1 \leq n \leq p_{j+1}^2 - 1)$ are $(p_j^2, p_j * p_{j+1}, p_j * p_{j+2}, p_j * p_{j+3}, \cdots\cdots, p_j * p_{j+E})$. $\quad (p_j * p_{j+E} < p_{j+1}^2)$ $\quad m(p_j)$ has the following composite pairs.

3.2.1 In $(p_j^2 + 1 \leq n = p_j^2 + k_0 \leq p_j * p_{j+1} - 1)$ $m(p_j)$ has the following 2 cases. $(k_0 = 1, 3, 5, \cdots\cdots, p_j * p_{j+1} - p_j^2 - 1)$ Because $m(p_j)$ has (p_j^2, k_0) and (k_0, p_j^2).

Case 5 : $\quad m(p_j) = 2 \quad$ (k_0 : 1 or prime number)

Case 6 : $\quad m(p_j) = 0 \quad$ (k_0 : composite number excluding 1)

3.2.2 In $(p_j * p_{j+1} + 1 \leq n = p_j * p_{j+1} + k_1 \leq p_j * p_{j+2} - 1)$ $m(p_j)$ has the following 2 cases. $(k_1 = 1, 3, 5, \cdots\cdots, p_j * p_{j+2} - p_j * p_{j+1} - 1)$ Because $m(p_j)$ has $(p_j * p_{j+1}, k_1)$ and $(k_1, p_j * p_{j+1})$.

Case 7 : $\quad 2 \leq m(p_j) \quad$ (k_1 : 1 or prime number)

Case 8 : $\quad 0 \leq m(p_j) \quad$ (k_1 : composite number excluding 1)

3.2.3 Similarly $m(p_j)$ has the following 2 cases in $(p_j * p_{j+2} + 1 \leq n = p_j * p_{j+2} + k_2 \leq p_j * p_{j+3} - 1)$.

Case 9 : $\quad 2 \leq m(p_j) \quad$ (k_2 : 1 or prime number)

Case 10 : $\quad 0 \leq m(p_j) \quad$ (k_2 : composite number excluding 1)

3.2.4 We can find that $m(p_j)$ has the following 2 cases from the above item 3.2.1—3.2.3 in $(p_j * p_{j+e} + 1 \leq n = p_j * p_{j+e} + k_e \leq p_j * p_{j+e+1} - 1)$ or $(p_j * p_{j+E} + 1 \leq n = p_j * p_{j+E} + k_E \leq p_{j+1}^2 - 1)$. $\quad (0 \leq e \leq E - 1)$

Case A : $\quad 2 \leq m(p_j) \quad$ (k_e or k_E : 1 or prime number)

Case B : $\quad 0 \leq m(p_j) \quad$ (k_e or k_E : composite number excluding 1)

If we assume Case A under the conditions of Case B, this assumption overestimates $m(p_j)$ and it underestimates $l(n)$ from (5) and (6). So it does not affect the proof of (3). Then we have the following (14).

$$2 \leq m(p_j) \qquad\qquad (p_j^2 + 1 \leq n \leq p_{j+1}^2 - 1) \tag{14}$$

3.3 We have the following (15) from (5), (6), (13).
The 1st " $\leq$ " in (15) holds true by ignoring $(0,2)_{p_j|n,(3\sim p_{j-1})|(n-p_j)}$. Because this

term can have only the value of 0 or 2.
The 2nd " $=$ " in (15) holds true from (5).
The 3rd " $=$ " in (15) holds true from (6).
The 2nd " $\leq$ " in (15) holds true from $\lfloor x \rfloor \leq x$.

$$\begin{aligned}
&m(p_j) = (1,2)_{p_j|n} * \lfloor \{n/2 - m(3) - m(5) - m(7) - \cdots\cdots - m(p_{j-2}) - m(p_{j-1})\}/p_j + 0.5 \rfloor \\
&- (0,2)_{p_j|n,(3\sim p_{j-1})|(n-p_j)} \\
&\leq (1,2)_{p_j|n} * \lfloor \{n/2 - m(3) - m(5) - m(7) - \cdots\cdots - m(p_{j-2}) - m(p_{j-1})\}/p_j + 0.5 \rfloor \\
&= (1,2)_{p_j|n} * \lfloor \{n/2 - M(n) + m(p_j) + (0,2)_{n-1}\}/p_j + 0.5 \rfloor \\
&= (1,2)_{p_j|n} * \lfloor \{l(n) + m(p_j) + (0,2)_{n-1}\}/p_j + 0.5 \rfloor \\
&\leq (1,2)_{p_j|n} * [\{l(n) + m(p_j) + (0,2)_{n-1}\}/p_j + 0.5] \\
&= (1,2)_{p_j|n} * \{l(n) + m(p_j) + (0,2)_{n-1}\}/p_j + (1,2)_{p_j|n}/2
\end{aligned} \tag{15}$$

We have the following (16) and (17) from (14), (15) and $(0,2)_{n-1} \leq 2$.

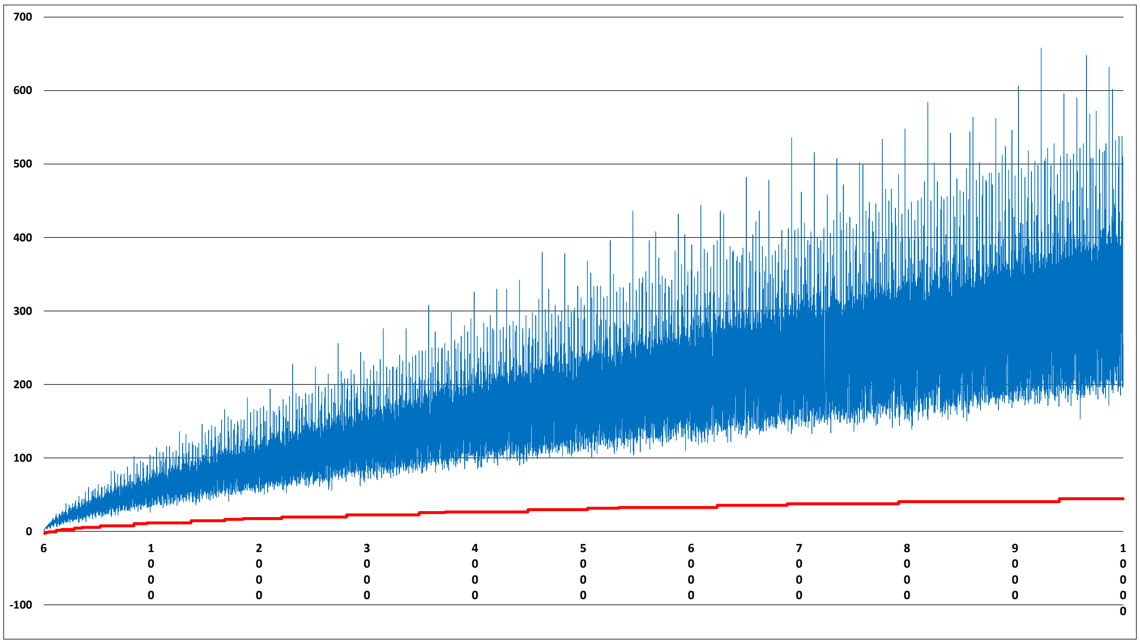
$$\begin{aligned}
l(n) &\geq \{p_j/(1,2)_{p_{j|n}}\} * m(p_j) - p_j/2 - m(p_j) - (0,2)_{n-1} \\
&= \{p_j/(1,2)_{p_{j|n}} - 1\} * m(p_j) - p_j/2 - (0,2)_{n-1} \\
&\geq \{p_j/(1,2)_{p_{j|n}} - 1\} * m(p_j) - p_j/2 - 2 \\
&\geq \{p_j/(1,2)_{p_{j|n}} - 1\} * 2 - p_j/2 - 2 \\
&= p_j/2 - 4 \qquad (p_j \nmid n \quad (1,2)_{p_{j|n}} = 2)
\end{aligned} \tag{16}$$

$$\begin{aligned}
l(n) &\geq \{p_j/(1,2)_{p_{j|n}} - 1\} * 2 - p_j/2 - 2 \\
&= 3 * p_j/2 - 4 > p_j/2 - 4 \qquad (p_j | n \quad (1,2)_{p_{j|n}} = 1)
\end{aligned} \tag{17}$$

We have the following (18) from (16) and (17).

$$1 < p_j/2 - 4 \leq l(n) \qquad (4 \leq j \quad p_4^2 + 1 = 11^2 + 1 = 122 \leq n) \tag{18}$$

The following (Graph 1) shows $l(n)$ and $(p_j/2 - 4)$. p_j is the largest prime number that satisfies $p < \sqrt{n}$.



Graph 1 ： $l(n)$(blue line)[1] and $\{p_j/2-4\}$(red line) from $n=6$ to $n=10,000$

3.4 We can find that $l(n)$ has the following properties from the above (18).

3.4.1 We have the following (19).

$$1 < l(n) \qquad\qquad (122 \leq n) \tag{19}$$

3.4.2 $l(n)$ diverges to ∞ with $n \to \infty$ because p_j diverges to ∞ with $j \to \infty$ i.e. $n \to \infty$.

4. Conclusion

Goldbach conjecture is true from the following item 4.1 and 4.2.

4.1 Goldbach conjecture is true in $122 \leq n$ from (3) and the above (19).

4.2 Goldbach conjecture is already confirmed to be true up to $n = 4 * 10^{18}$.

Appendix 1. : Investigation of $l(n)$

1.1 When an even number n is divided into 2 odd numbers x and y, we can find the pair of $\pi(n), l(n), m_{xx}, m_x, m_y$ and m_{xy} in $n/2$ pairs of (x, y) as shown in the following (Figure 3).

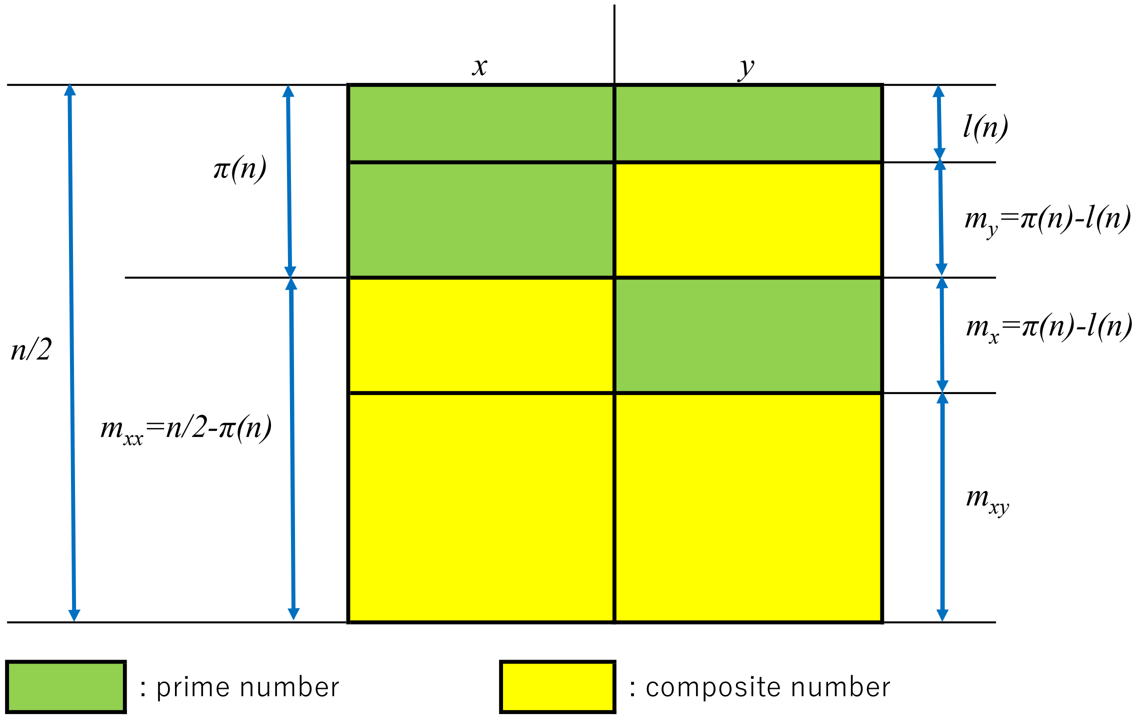


Figure 3 : Various pairs in $n/2$ pairs of (x, y)

We define as follows.

$\pi(n)$: $\pi(n)$ shows the total number of prime numbers which exist between 1 and n. But we use $\pi(n)$ in the above (Figure 3) for the total number of prime numbers which exist in $n/2$ odd numbers of (1, 3, 5, $\cdots\cdots$, $n-5, n-3, n-1$). Strictly speaking, this value must be $\pi(n-1)-1$. But we can say $\pi(n-1)-1 = \pi(n)-1 \fallingdotseq \pi(n)$ because n is an even number and a large number as shown in (4).

m_{xx} : the total number of pairs where x is a composite number. 1 is regarded as a composite number.

m_x : the total number of pairs where x and y are composite number and prime number respectively

1.2 We have the following (20) from Prime number theorem.

$$\frac{\pi(n)}{n} \sim \frac{n/\log n}{n} = \frac{1}{\log n} \qquad (n \to \infty) \tag{20}$$

We have $\lim_{n\to\infty}\dfrac{\pi(n)}{n} = \lim_{n\to\infty}\dfrac{1}{\log n} = 0$ from the above (20). Then we have the following (21) from (Figure 3) and $\lim_{n\to\infty}\dfrac{\pi(n)}{n} = 0$

$$m_{xx} = n/2 - \pi(n) = (n/2)\{1 - 2\pi(n)/n\} \quad \sim \quad n/2 \qquad (n \to \infty) \qquad (21)$$

When m_{xx} approaches $n/2$ with $n \to \infty$ as shown in the above (21), m_x approaches $\pi(n)$ with $n \to \infty$ due to the following reasons.

1.2.1 m_x shows the total number of prime numbers which exist in y of m_{xx} as shown in (Figure 3).

1.2.2 y of m_{xx} approaches $n/2$ odd numbers of $(1, 3, 5, \cdots\cdots, n-5, n-3, n-1)$ with $n \to \infty$ as shown in the above (21).

1.2.3 $(1, 3, 5, \cdots\cdots, n-5, n-3, n-1)$ has $\pi(n)$ prime numbers as shown in item 1.1.

Then we can have the following (22) from (Figure 3).

$$m_x = \pi(n) - l(n) = \pi(n)\{1 - l(n)/\pi(n)\} \quad \sim \quad \pi(n) \qquad (n \to \infty) \qquad (22)$$

We have $\lim_{n\to\infty}\dfrac{l(n)}{\pi(n)} = 0$ from the above (22). We have the following (23) from the above (21) and (22).

$$\frac{\pi(n) - l(n)}{n/2 - \pi(n)} \quad \sim \quad \frac{\pi(n)}{n/2} \qquad (n \to \infty) \qquad (23)$$

We have the following (24) from the above (23) and Prime number theorem.

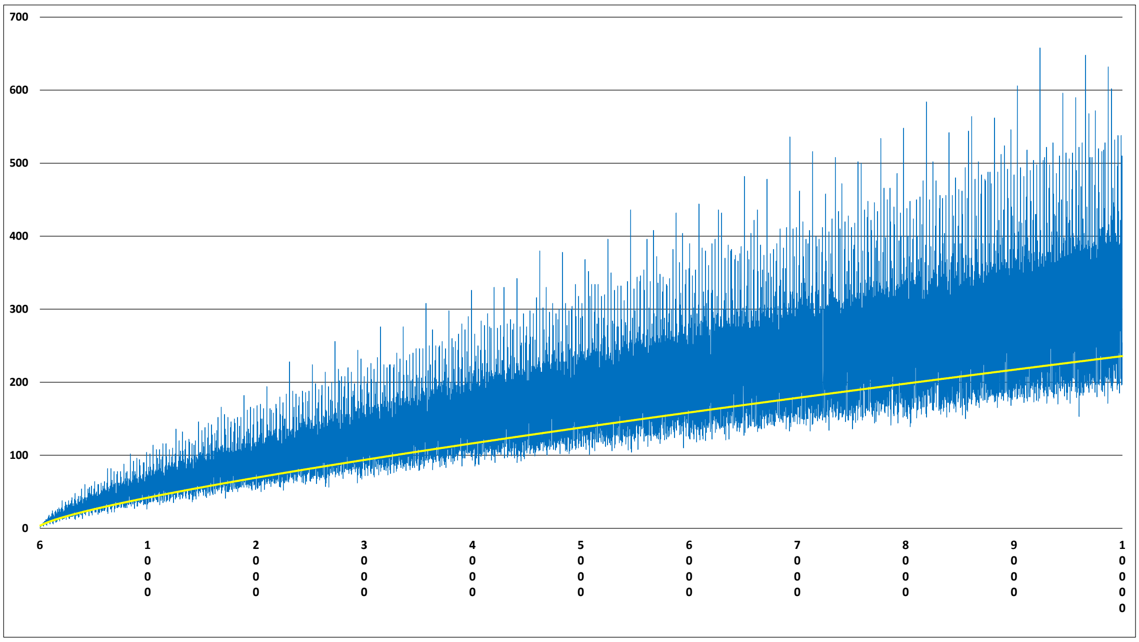
$$l(n) \quad \sim \quad \frac{\{\pi(n)\}^2}{n/2} \quad \sim \quad \frac{\{n/\log n\}^2}{n/2} = \frac{2n}{(\log n)^2} \qquad (n \to \infty) \qquad (24)$$

We can find that $l(n)$ has the following properties from the above (24).

1.2.4 $l(n)$ repeats increases and decreases with increase of n as shown in the following (Graph 2). But overall $l(n)$ is an increasing function regarding n because $\dfrac{2n}{(\log n)^2}$ is an increasing function regarding n.

1.2.5 $l(n)$ diverges to ∞ with $n \to \infty$ because $\dfrac{2n}{(\log n)^2}$ diverges to ∞ with $n \to \infty$.

1.3 $\dfrac{2n}{(\log n)^2}$ seems to approximate $l(n)$ sufficiently well as shown in the following (Graph 2).



Graph 2 : $l(n)$(blue line)[1] and $\dfrac{2n}{(\log n)^2}$(yellow line) from $n = 6$ to $n = 10,000$

References

[1] THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES

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