Proof of Riemann hypothesis

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process.

1. We make one identity regarding x from one equation that gives Riemann zeta function $\zeta(s)$ analytic continuation and 2 formulas ($1/2+a \pm bi$, $1/2-a \pm bi$) that show non-trivial zero point of $\zeta(s)$.
2. We find that the above identity holds only at $a = 0$.
3. Therefore non-trivial zero points of $\zeta(s)$ must be $1/2 \pm bi$ because a cannot have any value but zero.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $0 < Re(s)$. "$+\cdots\cdots$" means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots\cdots = (1 - 2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$. i is $\sqrt{-1}$.

$$S_0 = 1/2 + a \pm bi \qquad\qquad (0 \leq a < 1/2 \quad 14 < b) \tag{2}$$

The following (3) also shows non-trivial zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1 - S_0 = 1/2 - a \mp bi \tag{3}$$

We define the range of a and b as $0 \leq a < 1/2$ and $14 < b$ respectively. Then we can show all non-trivial zero points of $\zeta(s)$ by the above (2) and (3). Because non-trivial zero points of $\zeta(s)$ exist in the critical strip of $\zeta(s)$ $(0 < Re(s) < 1)$ and non-trivial zero points of $\zeta(s)$ found until now exist in the range of $14 < b$.
We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b \log 2)}{2^{1/2+a}} - \frac{\cos(b \log 3)}{3^{1/2+a}} + \frac{\cos(b \log 4)}{4^{1/2+a}} - \frac{\cos(b \log 5)}{5^{1/2+a}} + \cdots\cdots \tag{4}$$

$$0 = \frac{\sin(b \log 2)}{2^{1/2+a}} - \frac{\sin(b \log 3)}{3^{1/2+a}} + \frac{\sin(b \log 4)}{4^{1/2+a}} - \frac{\sin(b \log 5)}{5^{1/2+a}} + \cdots\cdots \tag{5}$$

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We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. The identity regarding x

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from the above (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from the above (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) regarding real number x from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. The value of (11) is always zero at any value of x.

$$\begin{aligned}
0 \equiv &\cos x\{\text{the right side of (9)}\} + \sin x\{\text{the right side of (10)}\} \\
= &\cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - \cdots\cdots\} \\
&+ \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - \cdots\cdots\} \\
= &f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) \\
&- f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \cdots\cdots
\end{aligned} \tag{11}$$

At $a = 0$ we have the following (8-1) and the above (11) holds at $a = 0$.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \equiv 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots \quad a = 0) \tag{8-1}$$

We have the following (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned}
0 = &f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1) \\
&- f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots
\end{aligned} \tag{12-1}$$

We have the following (12-2) by substituting $b\log 2$ for x in (11).

$$\begin{aligned}
0 = &f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2) + f(4)\cos(b\log 4 - b\log 2) \\
&- f(5)\cos(b\log 5 - b\log 2) + f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots
\end{aligned} \tag{12-2}$$

We have the following (12-3) by substituting $b\log 3$ for x in (11).

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3)\\ &- f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots \qquad (12\text{-}3)\end{aligned}$$

In the same way as above we can have the following (12-N) by substituting $b\log N$ for x in (11). $(N = 4, 5, 6, 7, \cdots\cdots)$

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N)\\ &- f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \cdots\cdots \qquad (12\text{-N})\end{aligned}$$

3. The solution for the identity of (11)

3.1 We define $g(k, N)$ as follows. $(k = 2, 3, 4, 5, \cdots\cdots \quad N = 1, 2, 3, 4, \cdots\cdots)$

$$\begin{aligned}g(k,N) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cos(b\log k - b\log 3) + \cdots + \cos(b\log k - b\log N)\\ &= \cos(b\log 1 - b\log k) + \cos(b\log 2 - b\log k) + \cos(b\log 3 - b\log k) + \cdots + \cos(b\log N - b\log k)\\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k) \qquad (13)\end{aligned}$$

We can have the following (14) from N equations of (12-1), (12-2), (12-3), $\cdots\cdots$, (12-N) with the method shown in item 1.4 of [Appendix 1].

$$\begin{aligned}0 =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cdots + \cos(b\log 2 - b\log N)\}\\ &-f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cdots + \cos(b\log 3 - b\log N)\}\\ &+f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cdots + \cos(b\log 4 - b\log N)\}\\ &-f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cdots + \cos(b\log 5 - b\log N)\}\\ &+\cdots\cdots\\ =& f(2)g(2, N) - f(3)g(3, N) + f(4)g(4, N) - f(5)g(5, N) + \cdots\cdots \qquad (14)\end{aligned}$$

3.2 If (11) holds, the sum of the right sides of infinite number equations of (12-1), (12-2), (12-3), (12-4), (12-5), $\cdots\cdots$ becomes zero. The rightmost side of (14) is the sum of the right sides of N equations of (12-1), (12-2), (12-3),$\cdots\cdots$, (12-N) as shown in item 1.4 of [Appendix 1]. Therefore if (11) holds, $\lim_{N\to\infty}$ {the rightmost side of (14)} $= 0$ must hold.
We have the following (22) in [Appendix 2 : Investigation of $g(k, N)$].

$$g(k, N) \quad \sim \quad \frac{N\cos(b\log N/k - \tan^{-1} b)}{\sqrt{1+b^2}}$$
$$(N \to \infty \quad k = 2, 3, 4, 5, \cdots\cdots \quad 0.477 * \pi < \tan^{-1} b < \pi/2) \qquad (22)$$

Here we define $F(a)$ as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + \cdots\cdots \qquad (15)$$

From (14), (22), $\cos(b\log N/2-\tan^{-1}b)\neq 0$ shown in [Appendix 3 : Proof of $\cos(b\log N/k-\tan^{-1}b)\neq 0$] and $\lim_{N\to\infty}\dfrac{\cos(b\log N/k-\tan^{-1}b)}{\cos(b\log N/2-\tan^{-1}b)}=1$ shown in [Appendix 4 : Proof of $\lim_{N\to\infty}\dfrac{\cos(b\log N/k-\tan^{-1}b)}{\cos(b\log N/2-\tan^{-1}b)}=1$] we have the following (16).

$$\begin{aligned}
&\{\text{the rightmost side of (14)}\}\\
&= f(2)g(2,N)-f(3)g(3,N)+f(4)g(4,N)-f(5)g(5,N)+\cdots\cdots\\
&\sim \quad f(2)\frac{N\cos(b\log N/2-\tan^{-1}b)}{\sqrt{1+b^2}}-f(3)\frac{N\cos(b\log N/3-\tan^{-1}b)}{\sqrt{1+b^2}}\\
&\quad +f(4)\frac{N\cos(b\log N/4-\tan^{-1}b)}{\sqrt{1+b^2}}-f(5)\frac{N\cos(b\log N/5-\tan^{-1}b)}{\sqrt{1+b^2}}+\cdots\cdots\\
&= \{\frac{N*\cos(b\log N/2-\tan^{-1}b)}{\sqrt{1+b^2}}\}\{f(2)-f(3)\frac{\cos(b\log N/3-\tan^{-1}b)}{\cos(b\log N/2-\tan^{-1}b)}\\
&\quad +f(4)\frac{\cos(b\log N/4-\tan^{-1}b)}{\cos(b\log N/2-\tan^{-1}b)}-f(5)\frac{\cos(b\log N/5-\tan^{-1}b)}{\cos(b\log N/2-\tan^{-1}b)}+\cdots\cdots\}\\
&\sim \quad \{\frac{N*\cos(b\log N/2-\tan^{-1}b)}{\sqrt{1+b^2}}\}*\{f(2)-f(3)+f(4)-f(5)+\cdots\cdots\}\\
&= \{\frac{N*\cos(b\log N/2-\tan^{-1}b)}{\sqrt{1+b^2}}\}*F(a) \qquad (N\to\infty) \qquad (16)
\end{aligned}$$

3.3 We have the following (17) from Leibniz's theorem[1]. Because $F(a)$ is an alternating series. $f(n)$ has a positive value that decreases and converges to zero with $n\to\infty$.

$$F(a)\geq f(2)-f(3)=(\frac{1}{2^{1/2-a}}-\frac{1}{2^{1/2+a}})-(\frac{1}{3^{1/2-a}}-\frac{1}{3^{1/2+a}})\geq 0 \qquad (17)$$

$\lim_{N\to\infty}\dfrac{N*\cos(b\log N/2-\tan^{-1}b)}{\sqrt{1+b^2}}$ diverges to $\pm\infty$. $0<F(a)$ holds in $0<a<1/2$ from the above (17). Then $\lim_{N\to\infty}\{\text{the rightmost side of (14)}\}$ diverges to $\pm\infty$ in $0<a<1/2$ from the above (16) i.e. (11) does not hold in $0<a<1/2$. (11) holds at $a=0$ as shown in section 2. Therefore the solution for the identity of (11) is only $a=0$.

4. Conclusion

a has the range of $0\leq a<1/2$ by the critical strip of $\zeta(s)$. However, a cannot have any value but zero as shown in the above section 3. Therefore non-trivial zero point of Riemann zeta function $\zeta(s)$ shown by (2) and (3) is $1/2\pm bi$ and other non-trivial zero point does not exist.

Appendix 1. : Equation construction

We can construct (9), (10), (11) and (14) by applying the following Theorem 1[1].

Theorem 1

If the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) hold.

$$\text{(Series 1)} = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = A$$
$$\text{(Series 2)} = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots\cdots = B$$
$$\text{(Series 3)} = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots\cdots = A + B$$
$$\text{(Series 4)} = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \cdots\cdots = A - B$$

1.1. Construction of (9)

We can have (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

1.2. Construction of (10)

We can have (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

1.3. Construction of (11)

We can have (11) as (Series 3) by regarding the following (11-1) and (11-2) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \cos x\{\text{the right side of (9)}\} \equiv 0 \tag{11-1}$$
$$\text{(Series 2)} = \sin x\{\text{the right side of (10)}\} \equiv 0 \tag{11-2}$$

1.4. Construction of (14)

1.4.1 We can have the following (12-1*2) as (Series 3) by regarding the following (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 1)} =& f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\ &+ f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\ &+ f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots = 0\end{aligned} \tag{12-1}$$

$$\begin{aligned}\text{(Series 2)} =& f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2)\\ &+ f(4)\cos(b\log 4 - b\log 2) - f(5)\cos(b\log 5 - b\log 2)\\ &+ f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots = 0\end{aligned} \tag{12-2}$$

$$\begin{aligned}\text{(Series 3)} =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2)\}\\ &+ \cdots\cdots = 0 + 0\end{aligned} \tag{12-1*2}$$

1.4.2 We can have the following (12-1*3) as (Series 3) by regarding the above (12-1*2) and the following (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 2)} = {} & f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3)\\ & + f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3)\\ & + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots = 0 \end{aligned} \tag{12-3}$$

$$\begin{aligned}&\text{(Series 3)}\\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3)\}\\ &\quad + \cdots\cdots = 0 + 0 \end{aligned} \tag{12-1*3}$$

1.4.3 We can have the following (12-1*4) as (Series 3) by regarding the above (12-1*3) and the following (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 2)} = {} & f(2)\cos(b\log 2 - b\log 4) - f(3)\cos(b\log 3 - b\log 4)\\ & + f(4)\cos(b\log 4 - b\log 4) - f(5)\cos(b\log 5 - b\log 4)\\ & + f(6)\cos(b\log 6 - b\log 4) - \cdots\cdots = 0 \end{aligned} \tag{12-4}$$

$$\begin{aligned}&\text{(Series 3)}\\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cos(b\log 2 - b\log 4)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cos(b\log 3 - b\log 4)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cos(b\log 4 - b\log 4)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cos(b\log 5 - b\log 4)\}\\ &\quad + \cdots\cdots = 0 + 0 \end{aligned} \tag{12-1*4}$$

1.4.4 In the same way as above we can have the following (12-1*N)=(14) as (Series 3) by regarding (12-1*N-1) and (12-N) as (Series 1) and (Series 2) respectively. ($N = 5, 6, 7, 8, \cdots\cdots$) $g(k, N)$ is defined in page 3. ($k = 2, 3, 4, 5, \cdots\cdots$)

$$\begin{aligned}&\text{(Series 3)}=\\ &f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cdots + \cos(b\log 2 - b\log N)\}\\ &-f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cdots + \cos(b\log 3 - b\log N)\}\\ &+f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cdots + \cos(b\log 4 - b\log N)\}\\ &-f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cdots + \cos(b\log 5 - b\log N)\}\\ &+\cdots\cdots\\ &\qquad = f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + \cdots\cdots\\ &\qquad = 0 + 0 \end{aligned} \tag{12-1*N}$$

Appendix 2. : Investigation of $g(k,N)$

2.1 We define G and H as follows. $(N = 1,2,3,4,\cdots\cdots)$

$$G = \lim_{N\to\infty}\frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\}$$
$$= \int_0^1 \cos(b\log x)dx \tag{20-1}$$
$$H = \lim_{N\to\infty}\frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\}$$
$$= \int_0^1 \sin(b\log x)dx \tag{20-2}$$

We calculate G and H by Integration by parts.

$$G = [x\cos(b\log x)]_0^1 + bH = 1 + bH \qquad H = [x\sin(b\log x)]_0^1 - bG = -bG$$

Then we can have the values of G and H from the above equations as follows.

$$G = \frac{1}{1+b^2} \qquad H = \frac{-b}{1+b^2} \tag{21}$$

2.2 From (13) and the above (21) we have the following (22).

$$g(k,N) = \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k)$$
$$= N\tfrac{1}{N}\{\cos(b\log\tfrac{1}{N}\tfrac{N}{k}) + \cos(b\log\tfrac{2}{N}\tfrac{N}{k}) + \cos(b\log\tfrac{3}{N}\tfrac{N}{k}) + \cdots + \cos(b\log\tfrac{N}{N}\tfrac{N}{k})\}$$
$$= N\tfrac{1}{N}\{\cos(b\log\tfrac{1}{N} + b\log\tfrac{N}{k}) + \cos(b\log\tfrac{2}{N} + b\log\tfrac{N}{k})$$
$$+\cos(b\log\tfrac{3}{N} + b\log\tfrac{N}{k}) + \cdots\cdots + \cos(b\log\tfrac{N}{N} + b\log\tfrac{N}{k})\}$$
$$= N\tfrac{1}{N}\cos(b\log\tfrac{N}{k})\{\cos(b\log\tfrac{1}{N}) + \cos(b\log\tfrac{2}{N}) + \cos(b\log\tfrac{3}{N}) + \cdots + \cos(b\log\tfrac{N}{N})\}$$
$$-N\tfrac{1}{N}\sin(b\log\tfrac{N}{k})\{\sin(b\log\tfrac{1}{N}) + \sin(b\log\tfrac{2}{N}) + \sin(b\log\tfrac{3}{N}) + \cdots + \sin(b\log\tfrac{N}{N})\}$$
$$\sim \qquad N\cos(b\log\frac{N}{k})G - N\sin(b\log\frac{N}{k})H$$
$$= N\cos(b\log\frac{N}{k})\frac{1}{1+b^2} + N\sin(b\log\frac{N}{k})\frac{b}{1+b^2}$$
$$= \frac{N}{\sqrt{1+b^2}}\{\cos(b\log\frac{N}{k})\frac{1}{\sqrt{1+b^2}} + \sin(b\log\frac{N}{k})\frac{b}{\sqrt{1+b^2}}\}$$
$$= \frac{N}{\sqrt{1+b^2}}\cos(b\log\frac{N}{k} - \tan^{-1}b)$$
$$(N\to\infty \qquad k = 2,3,4,5,\cdots\cdots \qquad 0.477*\pi < \tan^{-1}b < \pi/2) \tag{22}$$

Since the range of b is $14 < b$ as shown in section 1 in the text, the range of $\tan^{-1}b$ is $(0.477*\pi < \tan^{-1}b < \pi/2)$.

Appendix 3. : Proof of $\cos(b \log N/k - \tan^{-1} b) \neq 0$

For $\cos(b \log N/k - \tan^{-1} b) = 0$ to hold the following (23) must hold. We have the following (24) from (23).

$$b \log N/k - \tan^{-1} b = l * \pi/2 \qquad (k = 2, 3, 4, \cdots\cdots \quad l = \pm 1, \pm 3, \pm 5, \cdots\cdots) \tag{23}$$

$$N = k * e^{\frac{l*\pi/2 + tan^{-1} b}{b}} \tag{24}$$

$$l * \pi/2 + tan^{-1} b \neq 0 \qquad (0.477 * \pi < \tan^{-1} b < \pi/2 \quad l = \pm 1, \pm 3, \pm 5, \cdots\cdots) \tag{25}$$

Since the right side of (24) cannot be a natural number from (25), (24) does not hold. Therefore $\cos(b \log N/k - \tan^{-1} b) = 0$ does not hold.

Appendix 4. : Proof of $\lim_{N\to\infty} \frac{\cos(b \log N/k - \tan^{-1} b)}{\cos(b \log N/2 - \tan^{-1} b)} = 1$

4.1 We define as follows.

$$b \log N/2 - \tan^{-1} b = h(2, N) \tag{26}$$

$$b \log N/k - \tan^{-1} b = h(k, N) \tag{27}$$

We have the following (28) and (29) from (26) and (27).

$$\lim_{N\to\infty} h(2, N) = \infty \qquad \lim_{N\to\infty} h(k, N) = \infty \tag{28}$$

$$\lim_{N\to\infty} \frac{h(k, N)}{h(2, N)} = \lim_{N\to\infty} \frac{b \log N/k - \tan^{-1} b}{b \log N/2 - \tan^{-1} b} = 1 \tag{29}$$

4.2 We have the following (30) by performing Maclaurin expansion of $\cos x$. We have the following (31) from (30).

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\cdots \tag{30}$$

$$\sum_{n=0}^{n_0} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \sim \qquad \cos x \qquad (n_0 \to \infty) \tag{31}$$

4.3 We have the following (32) from (28) and (29).

$$\lim_{N\to\infty} \frac{1 - \frac{h(k,N)^2}{2!} + \frac{h(k,N)^4}{4!} - \frac{h(k,N)^6}{6!} + \cdots\cdots + (-1)^{n_0} \frac{h(k,N)^{2n_0}}{(2n_0)!}}{1 - \frac{h(2,N)^2}{2!} + \frac{h(2,N)^4}{4!} - \frac{h(2,N)^6}{6!} + \cdots\cdots + (-1)^{n_0} \frac{h(2,N)^{2n_0}}{(2n_0)!}}$$

$$= \lim_{N\to\infty} [\{\frac{h(k, N)}{h(2, N)}\}^{2n_0} \frac{h(k, N)^{-2n_0} - \frac{h(k,N)^{2-2n_0}}{2!} + \frac{h(k,N)^{4-2n_0}}{4!} - \frac{h(k,N)^{6-2n_0}}{6!} + \cdots\cdots + \frac{(-1)^{n_0}}{(2n_0)!}}{h(2, N)^{-2n_0} - \frac{h(2,N)^{2-2n_0}}{2!} + \frac{h(2,N)^{4-2n_0}}{4!} - \frac{h(2,N)^{6-2n_0}}{6!} + \cdots\cdots + \frac{(-1)^{n_0}}{(2n_0)!}}]$$

$$= \{1\} \frac{\frac{(-1)^{n_0}}{(2n_0)!}}{\frac{(-1)^{n_0}}{(2n_0)!}} = 1 \qquad (n_0 = 0, 1, 2, 3, \cdots\cdots \quad 1.12 * k < N) \tag{32}$$

If the following (33) holds true, we have the following (34) from $14 < b$ and (33). Then we will perform $N \to \infty$ in the above (32) in the range of $1.12 * k < N$ to avoid $h(k, N) = 0$.

$$b \log N/k - \tan^{-1} b = h(k, N) = 0 \tag{33}$$

$$1 < N/k < e^{\frac{\tan^{-1}(14)}{14}} = 1.11 \tag{34}$$

4.4 We can have the following (35-1) and (35-2) from (31) and (32).
Since (35-1) holds for any n_0 as shown in (32) and
$\frac{1-\frac{h(k,N)^2}{2!}+\frac{h(k,N)^4}{4!}-\frac{h(k,N)^6}{6!}+\cdots\cdots+(-1)^{n_0}\frac{h(k,N)^{2n_0}}{(2n_0)!}}{1-\frac{h(2,N)^2}{2!}+\frac{h(2,N)^4}{4!}-\frac{h(2,N)^6}{6!}+\cdots\cdots+(-1)^{n_0}\frac{h(2,N)^{2n_0}}{(2n_0)!}}$ approaches
$\frac{\cos(b \log N/k - \tan^{-1} b)}{\cos(b \log N/2 - \tan^{-1} b)}$ with $n_0 \to \infty$ from (31) as shown in (35-2),
$\lim_{N\to\infty} \frac{\cos(b \log N/k - \tan^{-1} b)}{\cos(b \log N/2 - \tan^{-1} b)} = 1$ holds true.

$$1 = \lim_{N\to\infty} \frac{1 - \frac{h(k,N)^2}{2!} + \frac{h(k,N)^4}{4!} - \frac{h(k,N)^6}{6!} + \cdots\cdots + (-1)^{n_0}\frac{h(k,N)^{2n_0}}{(2n_0)!}}{1 - \frac{h(2,N)^2}{2!} + \frac{h(2,N)^4}{4!} - \frac{h(2,N)^6}{6!} + \cdots\cdots + (-1)^{n_0}\frac{h(2,N)^{2n_0}}{(2n_0)!}} \tag{35-1}$$

$$\sim \quad \lim_{N\to\infty} \frac{\cos\{h(k, N)\}}{\cos\{h(2, N)\}} = \lim_{N\to\infty} \frac{\cos(b \log N/k - \tan^{-1} b)}{\cos(b \log N/2 - \tan^{-1} b)} \qquad (n_0 \to \infty) \tag{35-2}$$

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 28, (written in Japanese)

[2] Same as above, page 22

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