

# Proof of Riemann hypothesis

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Oct. 1. 2026

**Abstract.** This paper is a trial to prove Riemann hypothesis according to the following process.

1. We make one identity regarding  $x$  from one equation that gives Riemann zeta function  $\zeta(s)$  analytic continuation and 2 formulas  $(1/2 + a \pm bi, 1/2 - a \pm bi)$  that show non-trivial zero point of  $\zeta(s)$ .
2. We find that the above identity holds only at  $a = 0$ .
3. Therefore non-trivial zero points of  $\zeta(s)$  must be  $1/2 \pm bi$  because  $a$  cannot have any value but zero.

## 1. Introduction

The following (1) gives Riemann zeta function  $\zeta(s)$  analytic continuation for  $0 < \text{Re}(s)$ . “+.....” means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \dots = (1 - 2^{1-s})\zeta(s) \quad (1)$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of  $\zeta(s)$ .  $i$  is  $\sqrt{-1}$ .

$$S_0 = 1/2 + a \pm bi \quad (0 \leq a < 1/2 \quad 14 < b) \quad (2)$$

The following (3) also shows non-trivial zero point of  $\zeta(s)$  by the functional equation of  $\zeta(s)$ .

$$S_1 = 1 - S_0 = 1/2 - a \mp bi \quad (3)$$

We define the range of  $a$  and  $b$  as  $0 \leq a < 1/2$  and  $14 < b$  respectively. Then we can show all non-trivial zero points of  $\zeta(s)$  by the above (2) and (3). Because non-trivial zero points of  $\zeta(s)$  exist in the critical strip of  $\zeta(s)$  ( $0 < \text{Re}(s) < 1$ ) and non-trivial zero points of  $\zeta(s)$  found until now exist in the range of  $14 < b$ .

We have the following (4) and (5) by substituting  $S_0$  for  $s$  in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b \log 2)}{2^{1/2+a}} - \frac{\cos(b \log 3)}{3^{1/2+a}} + \frac{\cos(b \log 4)}{4^{1/2+a}} - \frac{\cos(b \log 5)}{5^{1/2+a}} + \dots \quad (4)$$

$$0 = \frac{\sin(b \log 2)}{2^{1/2+a}} - \frac{\sin(b \log 3)}{3^{1/2+a}} + \frac{\sin(b \log 4)}{4^{1/2+a}} - \frac{\sin(b \log 5)}{5^{1/2+a}} + \dots \quad (5)$$

We also have the following (6) and (7) by substituting  $S_1$  for  $s$  in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b \log 2)}{2^{1/2-a}} - \frac{\cos(b \log 3)}{3^{1/2-a}} + \frac{\cos(b \log 4)}{4^{1/2-a}} - \frac{\cos(b \log 5)}{5^{1/2-a}} + \dots \quad (6)$$

$$0 = \frac{\sin(b \log 2)}{2^{1/2-a}} - \frac{\sin(b \log 3)}{3^{1/2-a}} + \frac{\sin(b \log 4)}{4^{1/2-a}} - \frac{\sin(b \log 5)}{5^{1/2-a}} + \dots \quad (7)$$

## 2. The identity regarding $x$

We define  $f(n)$  as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \quad (n = 2, 3, 4, 5, \dots) \quad (8)$$

We have the following (9) from the above (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2) \cos(b \log 2) - f(3) \cos(b \log 3) + f(4) \cos(b \log 4) - f(5) \cos(b \log 5) + \dots \quad (9)$$

We also have the following (10) from the above (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2) \sin(b \log 2) - f(3) \sin(b \log 3) + f(4) \sin(b \log 4) - f(5) \sin(b \log 5) + \dots \quad (10)$$

We can have the following (11) regarding real number  $x$  from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. The value of (11) is always zero at any value of  $x$ .

$$\begin{aligned} 0 &\equiv \cos x \{\text{the right side of (9)}\} + \sin x \{\text{the right side of (10)}\} \\ &= \cos x \{f(2) \cos(b \log 2) - f(3) \cos(b \log 3) + f(4) \cos(b \log 4) - \dots\} \\ &\quad + \sin x \{f(2) \sin(b \log 2) - f(3) \sin(b \log 3) + f(4) \sin(b \log 4) - \dots\} \\ &= f(2) \cos(b \log 2 - x) - f(3) \cos(b \log 3 - x) + f(4) \cos(b \log 4 - x) \\ &\quad - f(5) \cos(b \log 5 - x) + f(6) \cos(b \log 6 - x) - \dots \end{aligned} \quad (11)$$

At  $a = 0$  we have the following (8-1) and the above (11) holds at  $a = 0$ .

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \equiv 0 \quad (n = 2, 3, 4, 5, \dots \quad a = 0) \quad (8-1)$$

We have the following (12-1) by substituting  $b \log 1$  for  $x$  in (11).

$$\begin{aligned} 0 &= f(2) \cos(b \log 2 - b \log 1) - f(3) \cos(b \log 3 - b \log 1) + f(4) \cos(b \log 4 - b \log 1) \\ &\quad - f(5) \cos(b \log 5 - b \log 1) + f(6) \cos(b \log 6 - b \log 1) - \dots \end{aligned} \quad (12-1)$$

We have the following (12-2) by substituting  $b \log 2$  for  $x$  in (11).

$$\begin{aligned} 0 &= f(2) \cos(b \log 2 - b \log 2) - f(3) \cos(b \log 3 - b \log 2) + f(4) \cos(b \log 4 - b \log 2) \\ &\quad - f(5) \cos(b \log 5 - b \log 2) + f(6) \cos(b \log 6 - b \log 2) - \dots \end{aligned} \quad (12-2)$$

We have the following (12-3) by substituting  $b \log 3$  for  $x$  in (11).

$$0 = f(2) \cos(b \log 2 - b \log 3) - f(3) \cos(b \log 3 - b \log 3) + f(4) \cos(b \log 4 - b \log 3) \\ - f(5) \cos(b \log 5 - b \log 3) + f(6) \cos(b \log 6 - b \log 3) - \dots \quad (12-3)$$

In the same way as above we can have the following (12-N) by substituting  $b \log N$  for  $x$  in (11). ( $N = 4, 5, 6, 7, \dots$ )

$$0 = f(2) \cos(b \log 2 - b \log N) - f(3) \cos(b \log 3 - b \log N) + f(4) \cos(b \log 4 - b \log N) \\ - f(5) \cos(b \log 5 - b \log N) + f(6) \cos(b \log 6 - b \log N) - \dots \quad (12-N)$$

### 3. The solution for the identity of (11)

3.1 We define  $g(n, N)$  as follows. ( $n = 2, 3, 4, 5, \dots$   $N = 1, 2, 3, 4, \dots$ )

$$g(n, N) = \cos(b \log n - b \log 1) + \cos(b \log n - b \log 2) + \cos(b \log n - b \log 3) + \dots + \cos(b \log n - b \log N) \\ = \cos(b \log 1 - b \log n) + \cos(b \log 2 - b \log n) + \cos(b \log 3 - b \log n) + \dots + \cos(b \log N - b \log n) \\ = \cos(b \log 1/n) + \cos(b \log 2/n) + \cos(b \log 3/n) + \dots + \cos(b \log N/n) \quad (13)$$

We can have the following (14) from  $N$  equations of (12-1), (12-2), (12-3),  $\dots$ , (12-N) with the method shown in item 1.4 of [Appendix 1].

$$0 = f(2) \{ \cos(b \log 2 - b \log 1) + \cos(b \log 2 - b \log 2) + \cos(b \log 2 - b \log 3) + \dots + \cos(b \log 2 - b \log N) \} \\ - f(3) \{ \cos(b \log 3 - b \log 1) + \cos(b \log 3 - b \log 2) + \cos(b \log 3 - b \log 3) + \dots + \cos(b \log 3 - b \log N) \} \\ + f(4) \{ \cos(b \log 4 - b \log 1) + \cos(b \log 4 - b \log 2) + \cos(b \log 4 - b \log 3) + \dots + \cos(b \log 4 - b \log N) \} \\ - f(5) \{ \cos(b \log 5 - b \log 1) + \cos(b \log 5 - b \log 2) + \cos(b \log 5 - b \log 3) + \dots + \cos(b \log 5 - b \log N) \} \\ + \dots \\ = f(2)g(2, N) - f(3)g(3, N) + f(4)g(4, N) - f(5)g(5, N) + \dots \quad (14)$$

3.2 If (11) holds, the sum of the right sides of infinite number equations of (12-1), (12-2), (12-3), (12-4), (12-5),  $\dots$  becomes zero. The rightmost side of (14) is the sum of the right sides of  $N$  equations of (12-1), (12-2), (12-3),  $\dots$ , (12-N) as shown in item 1.4 of [Appendix 1]. Therefore if (11) holds,  $\lim_{N \rightarrow \infty} \{ \text{the rightmost side of (14)} \} = 0$  must hold. Since (11) holds at  $a = 0$  as shown in section 2, hereafter we will investigate if (11) holds in  $0 < a < 1/2$ .

We have the following (22) in [Appendix 2 : Investigation of  $g(n, N)$ ].

$$g(n, N) \sim \frac{N \cos(b \log N/n - \tan^{-1} b)}{\sqrt{1 + b^2}} \\ (N \rightarrow \infty \quad n = 2, 3, 4, 5, \dots \quad 0.477 * \pi < \tan^{-1} b < \pi/2) \quad (22)$$

We have the following (29) in [Appendix 3 : Investigation of  $\cos(b \log N/n -$

$\tan^{-1} b)]$ .

$$\begin{aligned} \cos(b \log N/n - \tan^{-1} b) &\sim \cos(b \log N/2 - \tan^{-1} b) \\ (N \rightarrow \infty \quad n = 3, 4, 5, \dots) \end{aligned} \quad (29)$$

Here we define  $F(a)$  as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + \dots \quad (15)$$

From the above (14), (22) and (29) we have the following (16).

$$\begin{aligned} &\text{the rightmost side of (14)} \\ &= f(2)g(2, N) - f(3)g(3, N) + f(4)g(4, N) - f(5)g(5, N) + \dots \\ &\sim f(2) \frac{N \cos(b \log N/2 - \tan^{-1} b)}{\sqrt{1+b^2}} - f(3) \frac{N \cos(b \log N/3 - \tan^{-1} b)}{\sqrt{1+b^2}} \\ &\quad + f(4) \frac{N \cos(b \log N/4 - \tan^{-1} b)}{\sqrt{1+b^2}} - f(5) \frac{N \cos(b \log N/5 - \tan^{-1} b)}{\sqrt{1+b^2}} + \dots \\ &\sim f(2) \frac{N \cos(b \log N/2 - \tan^{-1} b)}{\sqrt{1+b^2}} - f(3) \frac{N \cos(b \log N/2 - \tan^{-1} b)}{\sqrt{1+b^2}} \\ &\quad + f(4) \frac{N \cos(b \log N/2 - \tan^{-1} b)}{\sqrt{1+b^2}} - f(5) \frac{N \cos(b \log N/2 - \tan^{-1} b)}{\sqrt{1+b^2}} + \dots \\ &= \frac{N * \cos(b \log N/2 - \tan^{-1} b)}{\sqrt{1+b^2}} * \{f(2) - f(3) + f(4) - f(5) + \dots\} \\ &= \frac{N * \cos(b \log N/2 - \tan^{-1} b)}{\sqrt{1+b^2}} * F(a) \quad (N \rightarrow \infty) \end{aligned} \quad (16)$$

3.3  $F(a)$  is an alternating series.  $f(n)$  has a positive value that decreases and converges to zero with  $n \rightarrow \infty$ . Then we have the following (17) from Leibniz's theorem[1].

$$\begin{aligned} F(a) \geq f(2) - f(3) &= \left( \frac{1}{2^{1/2-a}} - \frac{1}{2^{1/2+a}} \right) - \left( \frac{1}{3^{1/2-a}} - \frac{1}{3^{1/2+a}} \right) > 0 \\ &\quad (0 < a < 1/2) \end{aligned} \quad (17)$$

$\lim_{N \rightarrow \infty} \frac{N * \cos(b \log N/2 - \tan^{-1} b)}{\sqrt{1+b^2}}$  diverges to  $\pm\infty$ .  $0 < F(a)$  holds in  $0 < a < 1/2$  from the above (17). Then  $\lim_{N \rightarrow \infty} \{\text{the rightmost side of (14)}\}$  diverges to  $\pm\infty$  in  $0 < a < 1/2$  from the above (16) i.e. (11) does not hold in  $0 < a < 1/2$ . (11) holds at  $a = 0$  as shown in section 2. Therefore the solution for the identity of (11) is only  $a = 0$ .

#### 4. Conclusion

$a$  has the range of  $0 \leq a < 1/2$  by the critical strip of  $\zeta(s)$ . However,  $a$  cannot have any value but zero as shown in the above section 3. Therefore non-trivial zero point of Riemann zeta function  $\zeta(s)$  shown by (2) and (3) is  $1/2 \pm bi$  and other non-trivial zero point does not exist.

## Appendix 1. : Equation construction

We can construct (9), (10), (11) and (14) by applying the following Theorem 1[1].

Theorem 1

If the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) hold.

$$(\text{Series 1}) = a_1 + a_2 + a_3 + a_4 + a_5 + \dots = A$$

$$(\text{Series 2}) = b_1 + b_2 + b_3 + b_4 + b_5 + \dots = B$$

$$(\text{Series 3}) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \dots = A + B$$

$$(\text{Series 4}) = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \dots = A - B$$

### 1.1. Construction of (9)

We can have (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

### 1.2. Construction of (10)

We can have (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

### 1.3. Construction of (11)

We can have (11) as (Series 3) by regarding the following (11-1) and (11-2) as (Series 1) and (Series 2) respectively.

$$(\text{Series 1}) = \cos x \{\text{the right side of (9)}\} \equiv 0 \quad (11-1)$$

$$(\text{Series 2}) = \sin x \{\text{the right side of (10)}\} \equiv 0 \quad (11-2)$$

### 1.4. Construction of (14)

1.4.1 We can have the following (12-1\*2) as (Series 3) by regarding the following (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned} (\text{Series 1}) = & f(2) \cos(b \log 2 - b \log 1) - f(3) \cos(b \log 3 - b \log 1) \\ & + f(4) \cos(b \log 4 - b \log 1) - f(5) \cos(b \log 5 - b \log 1) \\ & + f(6) \cos(b \log 6 - b \log 1) - \dots = 0 \end{aligned} \quad (12-1)$$

$$\begin{aligned} (\text{Series 2}) = & f(2) \cos(b \log 2 - b \log 2) - f(3) \cos(b \log 3 - b \log 2) \\ & + f(4) \cos(b \log 4 - b \log 2) - f(5) \cos(b \log 5 - b \log 2) \\ & + f(6) \cos(b \log 6 - b \log 2) - \dots = 0 \end{aligned} \quad (12-2)$$

$$\begin{aligned} (\text{Series 3}) = & f(2) \{ \cos(b \log 2 - b \log 1) + \cos(b \log 2 - b \log 2) \} \\ & - f(3) \{ \cos(b \log 3 - b \log 1) + \cos(b \log 3 - b \log 2) \} \\ & + f(4) \{ \cos(b \log 4 - b \log 1) + \cos(b \log 4 - b \log 2) \} \\ & - f(5) \{ \cos(b \log 5 - b \log 1) + \cos(b \log 5 - b \log 2) \} \\ & + \dots = 0 + 0 \end{aligned} \quad (12-1*2)$$

1.4.2 We can have the following (12-1\*3) as (Series 3) by regarding the above (12-1\*2) and the following (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned} (\text{Series 2}) &= f(2) \cos(b \log 2 - b \log 3) - f(3) \cos(b \log 3 - b \log 3) \\ &\quad + f(4) \cos(b \log 4 - b \log 3) - f(5) \cos(b \log 5 - b \log 3) \\ &\quad + f(6) \cos(b \log 6 - b \log 3) - \dots = 0 \end{aligned} \quad (12-3)$$

$$\begin{aligned} (\text{Series 3}) &= f(2) \{ \cos(b \log 2 - b \log 1) + \cos(b \log 2 - b \log 2) + \cos(b \log 2 - b \log 3) \} \\ &\quad - f(3) \{ \cos(b \log 3 - b \log 1) + \cos(b \log 3 - b \log 2) + \cos(b \log 3 - b \log 3) \} \\ &\quad + f(4) \{ \cos(b \log 4 - b \log 1) + \cos(b \log 4 - b \log 2) + \cos(b \log 4 - b \log 3) \} \\ &\quad - f(5) \{ \cos(b \log 5 - b \log 1) + \cos(b \log 5 - b \log 2) + \cos(b \log 5 - b \log 3) \} \\ &\quad + \dots = 0 + 0 \end{aligned} \quad (12-1*3)$$

1.4.3 We can have the following (12-1\*4) as (Series 3) by regarding the above (12-1\*3) and the following (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned} (\text{Series 2}) &= f(2) \cos(b \log 2 - b \log 4) - f(3) \cos(b \log 3 - b \log 4) \\ &\quad + f(4) \cos(b \log 4 - b \log 4) - f(5) \cos(b \log 5 - b \log 4) \\ &\quad + f(6) \cos(b \log 6 - b \log 4) - \dots = 0 \end{aligned} \quad (12-4)$$

$$\begin{aligned} (\text{Series 3}) &= f(2) \{ \cos(b \log 2 - b \log 1) + \cos(b \log 2 - b \log 2) + \cos(b \log 2 - b \log 3) + \cos(b \log 2 - b \log 4) \} \\ &\quad - f(3) \{ \cos(b \log 3 - b \log 1) + \cos(b \log 3 - b \log 2) + \cos(b \log 3 - b \log 3) + \cos(b \log 3 - b \log 4) \} \\ &\quad + f(4) \{ \cos(b \log 4 - b \log 1) + \cos(b \log 4 - b \log 2) + \cos(b \log 4 - b \log 3) + \cos(b \log 4 - b \log 4) \} \\ &\quad - f(5) \{ \cos(b \log 5 - b \log 1) + \cos(b \log 5 - b \log 2) + \cos(b \log 5 - b \log 3) + \cos(b \log 5 - b \log 4) \} \\ &\quad + \dots = 0 + 0 \end{aligned} \quad (12-1*4)$$

1.4.4 In the same way as above we can have the following (12-1\*N)=(14) as (Series 3) by regarding (12-1\*N-1) and (12-N) as (Series 1) and (Series 2) respectively. ( $N = 5, 6, 7, 8, \dots$ )  $g(n, N)$  is defined in page 3. ( $n = 2, 3, 4, 5, \dots$ )

$$\begin{aligned} (\text{Series 3}) &= \\ &f(2) \{ \cos(b \log 2 - b \log 1) + \cos(b \log 2 - b \log 2) + \cos(b \log 2 - b \log 3) + \dots + \cos(b \log 2 - b \log N) \} \\ &- f(3) \{ \cos(b \log 3 - b \log 1) + \cos(b \log 3 - b \log 2) + \cos(b \log 3 - b \log 3) + \dots + \cos(b \log 3 - b \log N) \} \\ &+ f(4) \{ \cos(b \log 4 - b \log 1) + \cos(b \log 4 - b \log 2) + \cos(b \log 4 - b \log 3) + \dots + \cos(b \log 4 - b \log N) \} \\ &- f(5) \{ \cos(b \log 5 - b \log 1) + \cos(b \log 5 - b \log 2) + \cos(b \log 5 - b \log 3) + \dots + \cos(b \log 5 - b \log N) \} \\ &+ \dots \\ &= f(2)g(2, N) - f(3)g(3, N) + f(4)g(4, N) - f(5)g(5, N) + \dots \\ &= 0 + 0 \end{aligned} \quad (12-1*N)$$

**Appendix 2. : Investigation of  $g(n, N)$** 

2.1 We define  $G$  and  $H$  as follows. ( $N = 1, 2, 3, 4, \dots$ )

$$\begin{aligned} G &= \lim_{N \rightarrow \infty} \frac{1}{N} \{ \cos(b \log \frac{1}{N}) + \cos(b \log \frac{2}{N}) + \cos(b \log \frac{3}{N}) + \dots + \cos(b \log \frac{N}{N}) \} \\ &= \int_0^1 \cos(b \log x) dx \end{aligned} \quad (20-1)$$

$$\begin{aligned} H &= \lim_{N \rightarrow \infty} \frac{1}{N} \{ \sin(b \log \frac{1}{N}) + \sin(b \log \frac{2}{N}) + \sin(b \log \frac{3}{N}) + \dots + \sin(b \log \frac{N}{N}) \} \\ &= \int_0^1 \sin(b \log x) dx \end{aligned} \quad (20-2)$$

We calculate  $G$  and  $H$  by Integration by parts.

$$G = [x \cos(b \log x)]_0^1 + bH = 1 + bH \quad H = [x \sin(b \log x)]_0^1 - bG = -bG$$

Then we can have the values of  $G$  and  $H$  from the above equations as follows.

$$G = \frac{1}{1+b^2} \quad H = \frac{-b}{1+b^2} \quad (21)$$

2.2 From (13) and the above (21) we have the following (22).

$$\begin{aligned} g(n, N) &= \cos(b \log 1/n) + \cos(b \log 2/n) + \cos(b \log 3/n) + \dots + \cos(b \log N/n) \\ &= N \frac{1}{N} \{ \cos(b \log \frac{1}{N} \frac{N}{n}) + \cos(b \log \frac{2}{N} \frac{N}{n}) + \cos(b \log \frac{3}{N} \frac{N}{n}) + \dots + \cos(b \log \frac{N}{N} \frac{N}{n}) \} \\ &= N \frac{1}{N} \{ \cos(b \log \frac{1}{N} + b \log \frac{N}{n}) + \cos(b \log \frac{2}{N} + b \log \frac{N}{n}) \\ &\quad + \cos(b \log \frac{3}{N} + b \log \frac{N}{n}) + \dots + \cos(b \log \frac{N}{N} + b \log \frac{N}{n}) \} \\ &= N \frac{1}{N} \cos(b \log \frac{N}{n}) \{ \cos(b \log \frac{1}{N}) + \cos(b \log \frac{2}{N}) + \cos(b \log \frac{3}{N}) + \dots + \cos(b \log \frac{N}{N}) \} \\ &\quad - N \frac{1}{N} \sin(b \log \frac{N}{n}) \{ \sin(b \log \frac{1}{N}) + \sin(b \log \frac{2}{N}) + \sin(b \log \frac{3}{N}) + \dots + \sin(b \log \frac{N}{N}) \} \\ &\sim N \cos(b \log \frac{N}{n}) G - N \sin(b \log \frac{N}{n}) H \\ &= N \cos(b \log \frac{N}{n}) \frac{1}{1+b^2} + N \sin(b \log \frac{N}{n}) \frac{b}{1+b^2} \\ &= \frac{N}{\sqrt{1+b^2}} \{ \cos(b \log \frac{N}{n}) \frac{1}{\sqrt{1+b^2}} + \sin(b \log \frac{N}{n}) \frac{b}{\sqrt{1+b^2}} \} \\ &= \frac{N}{\sqrt{1+b^2}} \cos(b \log \frac{N}{n} - \tan^{-1} b) \\ &\quad (N \rightarrow \infty \quad n = 2, 3, 4, 5, \dots \quad 0.477 * \pi < \tan^{-1} b < \pi/2) \end{aligned} \quad (22)$$

Since the range of  $b$  is  $14 < b$  as shown in section 1 in the text, the range of  $\tan^{-1} b$  is  $(0.477 * \pi < \tan^{-1} b < \pi/2)$ .

### Appendix 3. : Investigation of $\cos(b \log N/n - \tan^{-1} b)$

3.1 We define as follows.

$$b \log N/2 - \tan^{-1} b = h(2, N) \quad (23)$$

$$b \log N/n - \tan^{-1} b = h(n, N) \quad (24)$$

We have the following (25) and (26) by performing Maclaurin expansion of  $\cos h(2, N)$  and  $\cos h(n, N)$ .

$$\cos h(2, N) = 1 - \frac{h(2, N)^2}{2!} + \frac{h(2, N)^4}{4!} - \frac{h(2, N)^6}{6!} + \dots + (-1)^k \frac{h(2, N)^{2k}}{(2k)!} + \dots \quad (25)$$

$$\cos h(n, N) = 1 - \frac{h(n, N)^2}{2!} + \frac{h(n, N)^4}{4!} - \frac{h(n, N)^6}{6!} + \dots + (-1)^k \frac{h(n, N)^{2k}}{(2k)!} + \dots \quad (k = 1, 2, 3, \dots) \quad (26)$$

3.2 We have the following (27) from (23) and (24). We have the following (28) from (27).

$$\lim_{N \rightarrow \infty} \frac{h(n, N)}{h(2, N)} = \lim_{N \rightarrow \infty} \frac{b \log N/n - \tan^{-1} b}{b \log N/2 - \tan^{-1} b} = \lim_{N \rightarrow \infty} \frac{b \log N - b \log n - \tan^{-1} b}{b \log N - b \log 2 - \tan^{-1} b} = 1 \quad (27)$$

$$h(n, N) \sim h(2, N) \quad (N \rightarrow \infty \quad n = 3, 4, 5, \dots) \quad (28)$$

We have the following (29) from (25), (26) and (28).

$$\begin{aligned} \cos(b \log N/n - \tan^{-1} b) &= \cos h(n, N) \\ &= 1 - \frac{h(n, N)^2}{2!} + \frac{h(n, N)^4}{4!} - \frac{h(n, N)^6}{6!} + \dots + (-1)^k \frac{h(n, N)^{2k}}{(2k)!} + \dots \\ &\sim 1 - \frac{h(2, N)^2}{2!} + \frac{h(2, N)^4}{4!} - \frac{h(2, N)^6}{6!} + \dots + (-1)^k \frac{h(2, N)^{2k}}{(2k)!} + \dots \\ &= \cos h(2, N) = \cos(b \log N/2 - \tan^{-1} b) \quad (N \rightarrow \infty \quad n = 3, 4, 5, \dots) \end{aligned} \quad (29)$$

## References

- [1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 28, (written in Japanese)
- [2] Same as above, page 22

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